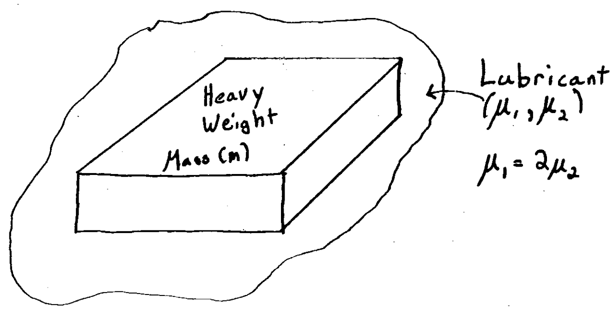
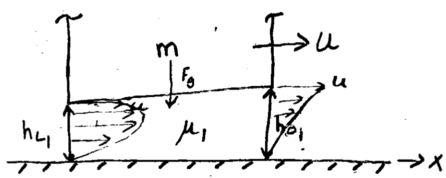
ME 351B – Answers to Homework #4

Problem 1: **(Cullen Buie)**





Assumptions:

1.) Combined Couette - Poiseuille flow
2.) Stokes Flow ← note: technically the lubrication approximation is not Stokes flow.
3.) $\mathcal{U}$ is constant
4.) $\frac{h_{o_1}}{h_{L_1}} \approx 1$

Stokes flow: $Re \ll 1$.
Lubrication: $Re \cdot \left(\frac{h}{L}\right)^2 \ll 1$ (see White p.187).
↑ this can occur even when Reynolds # is not small.

Solution:

Due to assumption ①, $\Delta p_{max} \propto \frac{\mu \mathcal{U} L}{h^2}$

We want to minimize Δp_{max} but we need to determine how h changes when fluid 1 with viscosity μ_1 is exchanged for the expensive fluid with viscosity $\mu_2 = \frac{1}{2}\mu_1$.

Δp_{max} is the same for both fluids since they both support the same mass (m).

$$\therefore \quad \frac{\mu_1 \mathcal{U} L}{h_1^2} = \frac{\mu_2 \mathcal{U} L}{h_2^2}$$

$$\frac{2\mu_2}{h_1^2} = \frac{\mu_2}{h_2^2}$$

$$2 = \frac{h_1^2}{h_2^2} \Rightarrow \frac{h_1}{h_2} = \sqrt{2} \qquad h_2 = \frac{h_1}{\sqrt{2}}$$

The force required to move the block is proportional to the shear stress at the interface between the lubricant and the mass.

$$\tau = \mu \frac{\partial u}{\partial y} = \frac{\mu \mathcal{U}}{h}$$ (using the Couette flow approximation w/negligible pressure gradient over the surface of the mass)

$$\frac{\tau_2}{\tau_1} = \frac{\frac{\mu_2 \mathcal{U}}{h_2}}{\frac{\mu_1 \mathcal{U}}{h_1}} = \frac{\frac{\cancel{\mu_1}}{2h_2}}{\frac{\cancel{\mu_1}}{h_1}} = \frac{h_1}{2h_2} = \frac{\sqrt{2}h_1}{2h_1} = \frac{\sqrt{2}}{2}$$

$$\therefore \tau_2 = \frac{\sqrt{2}}{2}\tau_1 = 0.707\,\tau_1$$

Using fluid 2 requires 70% of the force needed with fluid 1.

In other words:

- The force that one needs to exert on the body is proportional to the wall shear stress τ, which is obtained by approximating the flow as a linear Couette flow: $\tau = \mu U/h$.

- The gap height may depend on the viscosity μ of the fluid. To obtain this dependence, consider a force balance in the vertical direction: the weight of the object (a constant) must be balanced by the integrated pressure on the surface. This pressure is given by White's equation (3-240) and therefore scales like $\mu UL/h^2$, from which we deduce that $h \propto \sqrt{\mu}$.

- Therefore $\tau \propto \mu U/\sqrt{\mu}$ and $F \propto \sqrt{\mu}$.

Problem 2: **(Seongwon Kang)**

OK. But note that the viscosity μ is not really needed for the heat-transfer problem (cf eq. 2-50 in White). Without using μ, you would have obtaine $Pe = \frac{\rho C_p U D}{k}$ instead of Re and Pr.

If an incompressible flow.

2. Parameters

	heat transfer coefficient h	Length D	Conduction coef. K	ΔT	U_0	μ	ρ	C_p
unit	$MT^{-3}K^{-1}$	L	$MLT^{-3}K^{-1}$	K	LT^{-1}	$ML^{-1}T^{-1}$	ML^{-3}	$L^2T^{-2}K^{-1}$

M : mass, L : Length

T : time, K : Temperature

\# of parameter : 8

\# dimension : 4

$\therefore$ \# of Π : $8-4=4$

In this case I choose h, ρ, C_p, ΔT

$$\Pi_1 = h\, D^a K^b U^c \mu^d$$
$$= MT^{-3}K^{-1} L^a M^b L^b T^{-3b} K^{-b} L^c T^{-c} M^d L^{-d} T^{-d}$$

$$\begin{aligned} M &: 1+b+d=0 \\ L &: a+b+c-d=0 \\ T &: -3-3b-c-d=0 \\ K &: -1-b=0 \end{aligned} \Rightarrow \quad a=1,\ b=-1,\ c=0,\ d=0$$

$$\therefore \frac{hD}{K} = \frac{q''D}{K\Delta T} = Nu$$

$$\Pi_2 = \rho D^a k^b U^c \mu^d$$
$$= ML^{-3} L^a M^b L^b T^{-3b} K^{-b} L^c T^{-c} M^d L^{-d} T^{-d}$$

$$M : 1 + b + d = 0$$
$$L : -3 + a + b + c - d = 0$$
$$T : -3b - c - d = 0$$
$$K : -b = 0$$

$$\Rightarrow a = 1,\ b = 0,\ c = 1,\ d = -1$$

$$\therefore \frac{\rho U D}{\mu} = Re$$

$$\Pi_3 = c_p D^a k^b U^c \mu^d$$
$$= L^2 T^{-2} K^{-1} L^a M^b L^b T^{-3b} K^{-b} L^c T^{-c} M^d L^{-d} T^{-d}$$

$$M : b + d = 0$$
$$L : 2 + a + b + c - d = 0$$
$$T : -2 - 3b - c - d = 0$$
$$K : -1 - b = 0$$

$$\Rightarrow a = 0,\ b = -1,\ c = 0,\ d = 1$$

$$\therefore \frac{c_p \mu}{k} = Pr$$

$$\Pi_4 = \Delta T D^a k^b U^c c_p^d$$
$$= K L^a M^b L^b T^{-3b} K^{-b} L^c T^{-c} L^{2d} T^{-2d} K^{-d}$$

$$M : b = 0$$
$$L : a + b + c + 2d = 0$$
$$T : -3b - c - 2d = 0$$
$$K : 1 - b - d = 0$$

$$\Rightarrow a = 0,\ b = 0,\ c = -2,\ d = 1$$

$$\therefore \frac{\Delta T c_p}{U^2} \Rightarrow \frac{U^2}{c_p \Delta T} = Ec$$

Eckert No

In summary $Nu = f(Re, Pr, Ec)$

A few comments on Problem 2: The above dimensional analysis is very complete. It illustrates well the systematic procedure that one can follow to determine the dimensionless parameters from the dimensional variables. However it can be somewhat simplified:

- The first step of dimensional analysis is to determine which parameters are important for the situation at hand. This step, which may seem trivial, may also be the most important one: recognizing which variables are important and which can be discarded requires some physical intuition or knowledge of the governing equations, and can spare you a lot of work. In the example of problem 2, we were asked to consider the heat transfer problem in an incompressible viscous flow. In that case the viscous dissipation term in the energy equation is negligible (cf. equation (2-50) in White), and therefore the viscosity of the fluid is not needed. Repeating the above analysis without using the viscosity replaces the Reynolds and Prantl numbers by a single dimensionless parameter: the thermal Peclet number $Pe = \rho c_p UD/k$.

- Instead of considering the wall heat flux q and the temperature difference $\Delta T = T_w - T_\infty$ as two variables, one can use the heat transfer coefficient $q/\Delta T$ as a single parameter, in which case the Eckert number does not appear (the Eckert number is usually used in problems where the flux q is imposed instead of the temperature difference). Under these assumptions, we are left with two dimensionless parameters: $Nu = f(Pe)$.

Problem 3: **(Seongwon Kang)**

(a) Governing eq of heat transfer

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T \quad \text{(constant property)}$$

$$\Rightarrow \frac{DT}{Dt} = \alpha \nabla^2 T \ , \quad \text{where} \quad \alpha = \frac{k}{\rho C_p}$$

when non-dimensionalized using $T^* = \frac{T - T_0}{T_w - T_0}$, $t^* = \frac{U_0}{L} t$

$U^* = \frac{u}{U_0}$, $x_i^* = \frac{x_i}{L}$

$$\Rightarrow \frac{D^* T^*}{D^* t^*} = \frac{1}{Pr\, Re} \nabla^{*2} T^*$$

If $Pe = Pr\, Re$ is asymptotically small,

$$\Rightarrow \nabla^{*2} T^* = O(Re)$$
$$\approx 0$$

In cylindrical coords,

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

In this case $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0$ from symmetry

$$\therefore \nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) = 0 \quad \text{B.C)} \quad T(a) = T_w$$
$$T(R) = T_\infty$$

$$\Rightarrow \quad T(r) = C_1 \ln r + C_2$$

$$C_1 \ln a + C_2 = T_w$$

$$C_1 \ln K + C_2 = T_\infty$$

$$\Rightarrow \quad C_1 = \frac{T_\infty - T_w}{\ln K/a}, \qquad C_2 = T_w - \frac{T_\infty - T_w}{\ln K/a} \ln a$$

when $K \to \infty$, $C_1 \to 0$
$C_2 \to T_w$

$\therefore$ $T(r) \to T_w$ does not make sense.

One solution to this problem is Oseen's approximation

$$U \frac{\partial T}{\partial x} = \alpha \nabla^2 T \quad \Rightarrow \text{solution is not symmetric.}$$

In White P. 185, $Nu_m = B - \frac{P_r^2 R_e^2}{12}(16 + B^2)$

$$B = \frac{2}{\ln(8/P_r R_e) - \Upsilon}$$

Solving the 2D Laplace equation between two infinite concentric cylinders of radii a and b with respective temperatures T_a and T_∞ yields the following solution:

$$T(r) = T_a + (T_\infty - T_a)\frac{\log(r/a)}{\log(b/a)}$$

It is clear that letting $b \to \infty$ will lead to trouble... Doing it for a fixed value of r leads to $T(r) = T_a$ which obviously does not satisfy the BC at infinity...

Problem 4: **(Guillaume Blanquart)**

Now we consider again equation (*) but now in spherical coordinates. With the asumption of axisymmetrical solution $T=T(r)$, the equation becomes: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) = 0.$

$$\Leftrightarrow r^2 k \frac{\partial T}{\partial r} = A.$$

. Case 1: $k = k_0$ (constant).

$$(*) \Leftrightarrow r^2 \frac{\partial T}{\partial r} = A' \Leftrightarrow T = B - \frac{A'}{r}$$

$$\left.\begin{aligned} T_W &= B - \frac{A}{R} \\ T_\infty &= B \end{aligned}\right\} \Rightarrow \underline{T = T_\infty + (T_w - T_\infty)\frac{R}{r}}$$

We can compute the Nusselt number: $Nu = \dfrac{q_w R}{k_\infty (T_w - T_\infty)}$

$$q_W = k \left.\frac{\partial T}{\partial r}\right|_{r=R} = k_0 \frac{T_W - T_\infty}{R}.$$

then $\underline{Nu = 1}$.

Problem 5: **(Guillaume Blanquart)**

. The lubrication theory is based on the following assumptions:

$Re \frac{h^2}{L^2} \ll 1$ and $\frac{h}{L} \ll 1$.

and the final equation is: $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$.

But now, let's consider that the velocity undergoes a time dependant variation: $U = U(t)$. Then we need to keep some terms from the left and side:

$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$.

. As for every parameter, the left and the right hand sides have

~~s~~ of magnitude.

~~s~~ a consequence, the solution obtained from the Lubrication theory s valid by using $U(t)$ until the point the variation of U becomes big enough to interact with the right hand side of the equation:

$$\rho \frac{\partial u}{\partial t} \propto \rho \frac{U}{\tau}$$

$$\mu \frac{\partial^2 u}{\partial y^2} \propto \mu \frac{U}{h^2}$$

The limit time scale for velocity variation is:

$$\rho \frac{U}{\tau} \propto \mu \frac{U}{h^2} \iff \tau \propto \frac{\rho h^2}{\mu}.$$