ME 351B – Answers to Homework #4

Problem 1: **(Yaser Khalighi)**

for cylindrical coordinate:

u: streamwise

v: radial

* Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{1}{r}\frac{\partial}{\partial \theta}(u) = 0$$

$r = \delta Y + r_*$

$u = U U_*$

$v = V U_* \frac{\delta}{r_*}$

$\theta = X$

nondimentionalizing

$$\frac{1}{\cancel{\delta Y + r^*}}\frac{\partial}{\delta \partial Y}\left((\delta Y + r_*) V U_* \frac{\delta}{r_*}\right) + \frac{1}{\cancel{\delta Y + r^*}}\frac{\partial}{\partial X}(U U_*) = 0$$

δ is not a function of r!

$$\frac{\cancel{U_*}}{r_*}\frac{\partial}{\partial Y}\left((\delta Y + r_*) V\right) + \cancel{U_*}\frac{\partial}{\partial X}(U) = 0$$

$$\Rightarrow \frac{1}{r_*}\left(\delta V + (\delta Y + r^*)\frac{\partial V}{\partial Y}\right) + \frac{\partial}{\partial X} U = 0$$

$$\Rightarrow \underbrace{\frac{\delta}{r_*}}_{\varepsilon}\Big(Y \underbrace{\frac{\partial V}{\partial Y}}_{O(1)} + V\Big) + \frac{\partial V}{\partial Y} + \frac{\partial U}{\partial X} = 0$$

$$\Rightarrow \boxed{\frac{\partial V}{\partial Y} + \frac{\partial U}{\partial X} = O(\varepsilon)}$$

$$\varepsilon \to 0 \quad \frac{\partial V}{\partial Y} + \frac{\partial U}{\partial X} = 0$$

so continuity equation transforms to cartesian continuity equation as $\varepsilon \to 0$

Streamwise (θ) momentum in steady state case:

$$v \frac{\partial u}{\partial r} + \frac{1}{r} u \frac{\partial u}{\partial \theta} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} \right.$$

I non-dimensionalize terms separately:

$$v \frac{\partial u}{\partial r} = V u_* \frac{\delta}{r_*} \frac{u_* \partial U}{\delta \partial Y} = \frac{u_*^2}{r_*} V \frac{\partial U}{\partial Y} \quad \checkmark \text{ I}$$

$$\frac{1}{r} u \frac{\partial u}{\partial \theta} = \frac{1}{r^* + \delta Y} u_*^2 U \frac{\partial U}{\partial X} = \frac{u_*^2}{r_*(1+O(\varepsilon))} U \frac{\partial U}{\partial X} \approx \frac{U_*^2}{r_*} U \frac{\partial U}{\partial X} \quad \checkmark \text{ II}$$

$$\frac{uv}{r} = \frac{u_* U u_* \frac{\delta}{r^*} V}{r^* + \delta Y} = \frac{u_*^2}{r_*(1+O(\varepsilon))} \varepsilon U V = O(\varepsilon) \approx 0 \quad \checkmark \text{ III}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} = -\frac{1}{\rho(r_* + \delta Y)} \frac{\partial}{\partial X}\left(\rho u_*^2 P + P_*\right) = \frac{u_*^2}{r_* + \delta Y}\frac{\partial P}{\partial X} = \frac{u_*^2}{r_*(1+O(\varepsilon))}\frac{\partial P}{\partial X}$$

$$\approx \frac{u_*^2}{r_*}\frac{\partial P}{\partial X} \quad \checkmark \text{ IV}$$

$$\nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} \right]$$

$$= \nu \left[\frac{u_*}{\delta^2}\frac{\partial^2 U}{\partial Y^2} + \frac{u_*}{(r_* + \delta Y)^2}\frac{\partial^2 U}{\partial X^2} + \frac{u_*}{r_* + \delta Y}\frac{\partial U}{\delta \partial Y} + \frac{2}{(r_* + \delta Y)^2}\frac{\delta u_*}{r_*}\frac{\partial V}{\partial X} - \frac{u_* U}{(r_* + \delta Y)^2} \right]$$

$$= \frac{u^{*2}}{r^*}\frac{1}{Re}\left[\left(\frac{r_*}{\delta}\right)^2 \frac{\partial^2 U}{\partial Y^2} + \frac{1}{(1+O(\varepsilon))^2}\frac{\partial^2 U}{\partial X^2} + \frac{1}{\varepsilon(1+O(\varepsilon))}\frac{\partial U}{\partial Y} + \frac{2\varepsilon}{(1+O(\varepsilon))^2}\frac{\partial V}{\partial X} - \frac{U}{(1+O(\varepsilon))^2} \right]$$

$$\approx \frac{u^{*2}}{r^*}\frac{1}{Re}\left(\frac{r_*}{\delta}\right)^2 \frac{\partial^2 U}{\partial Y^2} \quad \checkmark \text{ V}$$

$$\Rightarrow \mathrm{I} \ldots \mathrm{V} \Rightarrow \boxed{v\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = \frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{r*}{\delta}\right)^2 \frac{\partial^2 u}{\partial y^2} + O(\varepsilon)}$$

b) If we wat to make viscous term competitive in this thin layer

$$\frac{1}{Re}\left(\frac{r*}{\delta}\right)^2 = O(1) \Rightarrow \frac{\delta}{r^*} = \frac{O(1)}{\sqrt{Re}}$$

$$\Rightarrow \boxed{\delta = \frac{r^*}{\sqrt{Re}} O(1)}$$

c) I call δ, boundary layer thickness for cylinder.

Problem 2: **(Ed Knudsen)**

Steady laminar boundary layer solutions for a certain flow problem at 150 different Re # are desired. Geometry of problem is fixed but other parameters are not. How many solutions need to be obtained?

The steady laminar boundary layer problem can be non-dimensionalized, as is done in problem #1. Then, if δ is picked as $\delta = r^*/\sqrt{Re}$, the dimensionless problem ceases to be dependent on Re. Thus, the engineer could solve the problem once and the solution will be good for all Re #'s. However, to get back to a dimensional solution, the Re # is required, and so there are 150 different solutions that can be returned for 150 different flows, where each unique solution is obtained by inputing the Re # to the dimensionless solution.

Problem 3: (Ed Knudsen)

Find the inviscid (potential flow) solution for steady flow over a circular cylinder.

	without circulation	with circulation
streamlines:	$\psi = U\left(r-\frac{a^2}{r}\right)\sin\theta$	$\psi = U\left(r-\frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)$
$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$:	$u_r = U\left(1-\frac{a^2}{r^2}\right)\cos\theta$	$u_r = U\left(1-\frac{a^2}{r^2}\right)\cos\theta$
$u_\theta = \frac{-\partial\psi}{\partial r}$:	$u_\theta = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta$	$u_\theta = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}$
$u^2 = u_r^2 + u_\theta^2$:	$u^2 = U^2\left[\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta + \left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta\right]$	$u^2 = U^2\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta + \left[U\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right]^2$

The pressure is $p - p_\infty = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho u^2$ (Bernoulli)

no circ: $$p-p_\infty = \frac{1}{2}\rho U^2\left[1-\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta - \left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta\right]$$

with circ: $$p-p_\infty = \frac{1}{2}\rho\left[U^2 - U^2\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta - \left(-U\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right)^2\right]$$

Find $\frac{\partial p}{r\partial\theta}$ and $\frac{\partial p}{\partial r}$:

no circ: $$\frac{\partial p}{\partial r} = \frac{1}{2}\rho U^2\left[2\left(1-\frac{a^2}{r^2}\right)\left(\frac{2a^2}{r^3}\right)\cos^2\theta - 2\left(1+\frac{a^2}{r^2}\right)\left(\frac{-2a^2}{r^3}\right)\sin^2\theta\right]$$

simplify: $$\frac{\partial p}{\partial r} = \frac{1}{2}\rho U^2\left[\frac{4a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta + \frac{4a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta\right]$$

$$\frac{\partial p}{\partial r} = \frac{2a^2}{r^3}\rho U^2\left[1-\frac{a^2}{r^2}\cos^2\theta + \frac{a^2}{r^2}\sin^2\theta\right]$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{1}{2r}\rho U^2\left[+\left(1-\frac{a^2}{r^2}\right)^2(2)\cos\theta\sin\theta - \left(1+\frac{a^2}{r^2}\right)^2(2)\sin\theta\cos\theta\right]$$

simplify: $$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{\sin(2\theta)}{2r}\rho U^2\left[\left(1-\frac{a^2}{r^2}\right)^2 - \left(1+\frac{a^2}{r^2}\right)^2\right]$$

with circ: $$\frac{\partial p}{\partial r} = \frac{1}{2}\rho\left[-U^2(2)\left(1-\frac{a^2}{r^2}\right)\left(\frac{2a^2}{r^3}\right)\cos^2\theta - 2\left(-U\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right)\left(-U\left(\frac{-2a^2}{r^3}\right)\sin\theta + \frac{\Gamma}{2\pi r^2}\right)\right]$$

simplify: $$\frac{\partial p}{\partial r} = \frac{1}{2}\rho\left[\frac{-4U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta + 2\left(U\left(1+\frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right)\left(\frac{2Ua^2}{r^3}\sin\theta + \frac{\Gamma}{2\pi r^2}\right)\right]$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{1}{2r}\rho\left[+U^2\left(1-\frac{a^2}{r^2}\right)^2(2)\cos\theta\sin\theta - 2\left(-U\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right)\left(-U\left(1+\frac{a^2}{r^2}\right)\cos\theta\right)\right]$$

simplify: $$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{1}{2r}\rho\left[U^2\left(1-\frac{a^2}{r^2}\right)^2\sin(2\theta) + 2\left(U\left(1+\frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right)\left(-U\left(1+\frac{a^2}{r^2}\right)\cos\theta\right)\right]$$

Evaluate the derivatives on the cylinder surface $(r=a)$.

no circ: $\left.\frac{\partial p}{\partial r}\right|_{r=a} = \frac{2}{a}\rho U^2\left[1-\cos^2\theta+\sin^2\theta\right] = \boxed{4\rho U^2\,\frac{\sin^2\theta}{a} = \left.\frac{\partial p}{\partial r}\right|_{r=a}}$

$\left.\frac{1}{r}\frac{\partial p}{\partial \theta}\right|_{r=a} = \frac{\sin(2\theta)}{2a}\rho U^2[-4] = \boxed{-4\rho U^2\,\frac{\sin\theta\cos\theta}{a} = \left.\frac{1}{r}\frac{\partial p}{\partial \theta}\right|_{r=a}}$

with circ: $\left.\frac{\partial p}{\partial r}\right|_{r=a} = \frac{1}{2}\rho\left[2\left(U(2)\sin\theta+\frac{\Gamma}{2\pi a}\right)\left(\frac{2U}{a}\sin\theta+\frac{\Gamma}{2\pi a^2}\right)\right]$

$= \boxed{\frac{\rho}{a}\left(2U\sin\theta+\frac{\Gamma}{2\pi a}\right)^2 = \left.\frac{\partial p}{\partial r}\right|_{r=a}}$ ✓

$\left.\frac{1}{r}\frac{\partial p}{\partial \theta}\right|_{r=a} = \frac{1}{2a}\rho\left[2\left(U(2)\sin\theta+\frac{\Gamma}{2\pi a}\right)\left(-U(2)\cos\theta\right)\right]$

$= \boxed{\frac{\rho}{a}\left[\left(2U\sin\theta+\frac{\Gamma}{2\pi a}\right)\left(-2U\cos\theta\right)\right] = \left.\frac{1}{r}\frac{\partial p}{\partial \theta}\right|_{r=a}}$ ✓

For both the circulation and no-circulation cases, the pressure derivatives, in the normal and tangent directions at the cylinder surface are of the same order of magnitude. ✓

The above derivation shows that (in dimensional variables) $\partial p/\partial r$ and $\partial p/r\partial\theta$ are of the same order of magnitude on the surface of the cylinder. However, non-dimensionalizing the variables using the transformation of Problem 1 shows that:

$$\frac{\partial P}{\partial Y} = O\left(\frac{\delta}{a}\frac{\partial P}{\partial X}\right)$$

and therefore $\partial P/\partial Y \ll \partial P/\partial X$.

Another way to look at this is that even though the transverse pressure gradient is large, it only creates very small pressure changes over the thickness δ of the boundary layer:

$$\Delta p_{transverse} \sim \delta \times \frac{\partial p}{\partial r} \ll \Delta p_{streamwise} \sim a \times \frac{\partial p}{a\partial\theta}$$

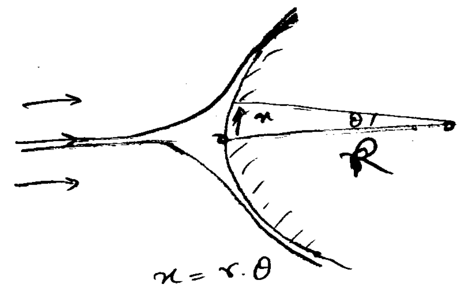
Problem 4: **(Tarun Khurana)**

$$U_e(x) = a x^m$$

$m = 1$ for B.L. at stagnation pt. of blunt body like cylinder

$$U_e(x) = a x$$

$$U_e(\theta) = a \cdot R\theta$$



For flow past cylinder

$$u_r = U_\infty \left(1 - \frac{R^2}{r^2}\right) \cos\theta$$

$$u_\theta = -U_\infty \left(1 + \frac{R^2}{r^2}\right) \sin\theta$$

at $r = R$

$$u_r = 0$$

$$u_\theta = 2 U_\infty \cdot \sin\theta.$$

for small values of θ

$$\sin\theta \sim \theta$$

$$u_\theta \simeq 2 U_\infty \theta. = a R.\theta.$$

$$\therefore \boxed{a = \frac{2 U_\infty}{R}}$$