Solution to Homework #8 (Yaser Khalighi)

Problem 1:

$$\frac{u}{u_e} = 2\eta - 2\eta^3 + \eta^4 \qquad \eta = \frac{y}{\delta}$$

$$u' = (2 - 6\eta^2 + 4\eta^3)u_e \times \frac{1}{\delta}$$

$$u'' = (-12\eta + 12\eta^2)u_e \times \frac{1}{\delta^2} = 12(\eta^2 - \eta)\frac{u_e}{\delta^2}$$

$$\begin{cases} (U-c)(v'' - \alpha^2 v) - U'' v = 0 \\ v(0) = 0 \\ v(\infty) = 0 \end{cases}$$

I don't know why White assumes $v'(0)=0$, it is a no slip boundary condition but it does not make any sense as we don't have viscousity term. indeed actually in page 346 as ~~equation~~ (5-22) he uses proper boundary condition.

In this problem I want to redo Rayleigh's approach (at least for warming up)

$$(U-c)(v'' - \alpha^2 v) - U'' v = 0$$

$$\bar{v} v'' - \alpha^2 v \bar{v} - \frac{U'' v \bar{v}}{U - c} = 0 \Rightarrow \bar{v} v'' - \alpha^2 |v|^2 - \frac{U'' |v|^2 \,[U - c_r + i c_i]}{[U - (c_r + i c_i)][U - c_r - i c_i]}$$

$$\Rightarrow \int_0^\infty \bar{v} v'' dy - \int_0^\infty \alpha^2 |v|^2 dy - \int_0^\infty \frac{U'' |v|^2 \,[U - c_r + i c_i]}{(U - c_r)^2 + (c_i)^2} dy = 0$$

$$-\int_0^\infty |v'|^2 dy + \cancel{\bar{v} v' \Big|_0^\infty}_{\,0} \Rightarrow \text{imaginary part: } \int_0^\infty \frac{c_i U'' |v|^2}{(U - c_r)^2 + (c_i)^2} dy = 0$$

$$U'' = \begin{cases} 12(\eta^2 - \eta)\dfrac{U_e}{\delta^2} & 0 \leq \eta \leq 1 \\ 0 & \eta > 1 \end{cases}$$

So $C_i = 0$ as U'' has no change in sign
i.e. U has no inflection point.

$$\Rightarrow \boxed{C = C_r}$$

$$v = v_r + i v_i$$

$$\begin{cases} (U - C_r)(v_r'' + i v_i'' - \alpha^2 v_r - i\alpha^2 v_i) - U''(v_r + i v_i) = 0 \\ v_r(0) = v_i(0) = 0 \\ v_r(\infty) = v_i(\infty) = 0 \end{cases}$$

$$\begin{cases} (U - C_r)(v_r'' - \alpha^2 v_r) - U'' v_r = 0 \\ (U - C_r)(v_i'' - \alpha^2 v_i) - U'' v_i = 0 \\ v_r(0) = v_i(0) = 0 \qquad v_r(\infty) = v_i(\infty) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (U - C_r)(v_r'' - \alpha^2 v_r) - U'' v_r = 0 \\ v_r(0) = v_r(\infty) = 0 \end{cases} + \{\text{the same for imaginary part.}$$

So solving the problem for each of real or imaginary part is exactly the same

for $\delta \leq y \leq 2\delta$ we have: $U'' = 0 \Rightarrow (U - C_r)(v_r'' - \alpha^2 v_r) = 0$

$$\Rightarrow v_r'' - \alpha^2 v_r = 0 \Rightarrow v_r = A_1 e^{\alpha y} + A_2 e^{-\alpha y}$$

$$A_1 = 0 \Rightarrow v_r = A_2 e^{-\alpha y}$$

$$\begin{cases} v_r\big|_{y=\delta} = A_2 e^{-\alpha\delta} \\ v_r'\big|_{y=\delta} = -\alpha A_2 e^{-\alpha\delta} \end{cases}$$

as we are interested in eigenvalue problems, we can multiply every thing by $\dfrac{1}{A_2 e^{-\alpha\delta}}$

$$\begin{cases} v_r(\delta) = 1 \\ v_r'(\delta) = -\alpha \end{cases}$$

So I solve this problem:

For specified α, find c_r such that the ODE problem:

$$\begin{cases} (U - c_r)(v_r'' - \alpha^2 v_r) - U'' v_r = 0 \\ v_r(\delta) = 1 \\ v_r'(\delta) = -\alpha \end{cases}$$

gives $v_r(0) = 0$.

lets first nondimensionalize the problem:

$$\begin{cases} \left(\frac{U}{U_e} - \frac{c_r}{U_e}\right)\left(\frac{v_r''}{U_e} - \alpha^2 \frac{v_r}{U_e}\right) - \left(\frac{U}{U_e}\right)'' \frac{v_r}{U_e} = 0 \\ \frac{v_r}{U_e}\left(\eta = \frac{y}{\delta} = 1\right) = 1 \\ \left(\frac{v}{U_e}\right)'\left(\eta = \frac{y}{\delta} = 1\right) = -\alpha \end{cases}$$

$\}\leftarrow$ as I solve the eigenvalue problem, multiplying by a factor is no problem.

$[\ ]' = \frac{d}{d\eta} = \frac{d}{dy} \cdot \frac{1}{\delta}$

$$\begin{cases} (U^* - c_r^*)\left(\frac{v_r^{*''}}{\delta^2} - \alpha^2 v_r^*\right) - \frac{U^{*''}}{\delta^2} v_r^* = 0 \\ v_r^*(1) = 1 \\ \frac{v_r^{*'}}{\delta}(1) = -\alpha \end{cases}$$

$$v_r^{*''} = \left(\frac{U^{*''}}{U^* - c_r^*} + (\alpha\delta)^2\right) v_r^*$$

$$\begin{cases} v_r^{*''} = \left(\frac{12\eta^2 - 12\eta}{2\eta - 2\eta^3 + \eta^4 - c_r^*} + (\alpha\delta)^2\right) v_r^* \\ v_r^*(1) = 1 \\ v_r^{*'}(1) = -\alpha\delta \end{cases}$$

So the problem reduces to:

$$\begin{cases} v_r^{*''} = \left[12 \dfrac{\eta^2 - \eta}{2\eta - 2\eta^3 + \eta^4 - C_r^*} + (\alpha\delta)^2 \right] v_r^* \\ v_r^*(1) = 1 \\ v_r^{*'}(1) = -\alpha\delta \end{cases}$$

for determined $\alpha\delta$, we should find c_r^* to give $v_r^*(0) = 0$.

the pseudo code I implement in MATLAB is:

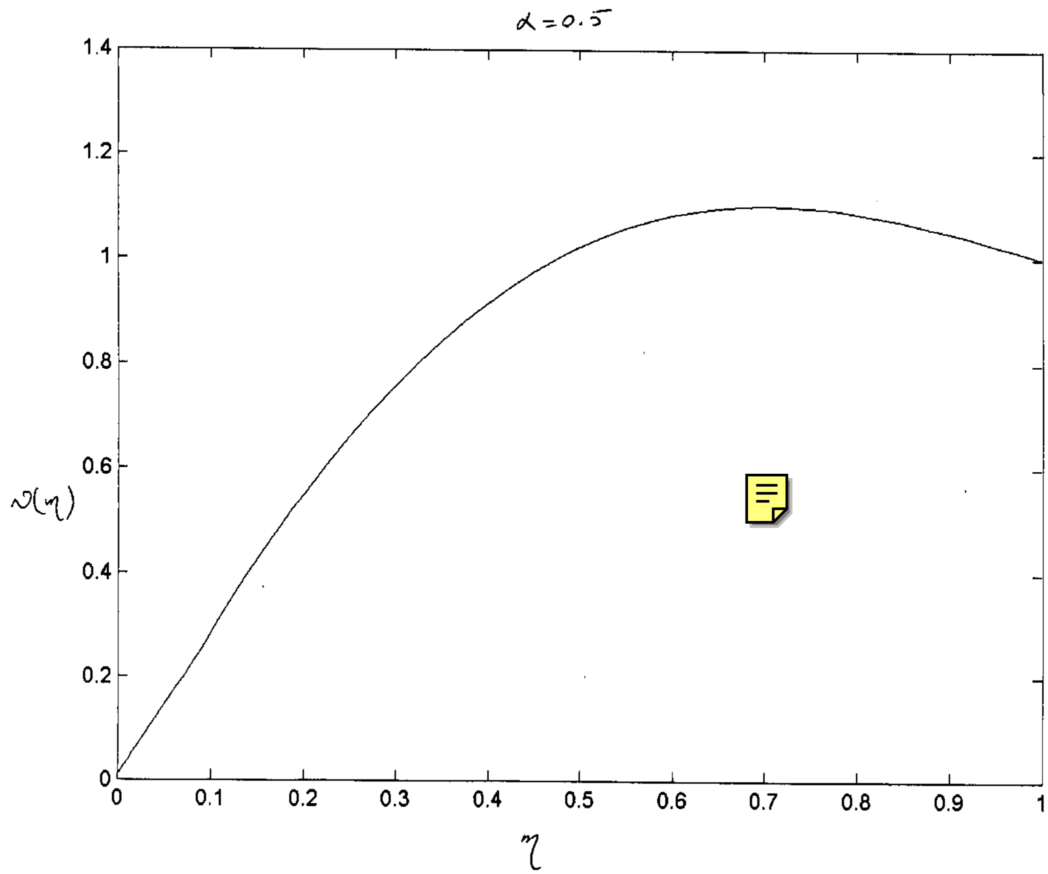
- guess C_r^*
- by Runge-kutta 4, solve $v_r^{*''} =$ ______
- find $v_r^*(0)$
- Using $v_r^*(0)$ and $v_r^*(0)$ in the last step, find new C_r^*

to guess New C_r^*. I use an systematic guess based on linear interpolation:

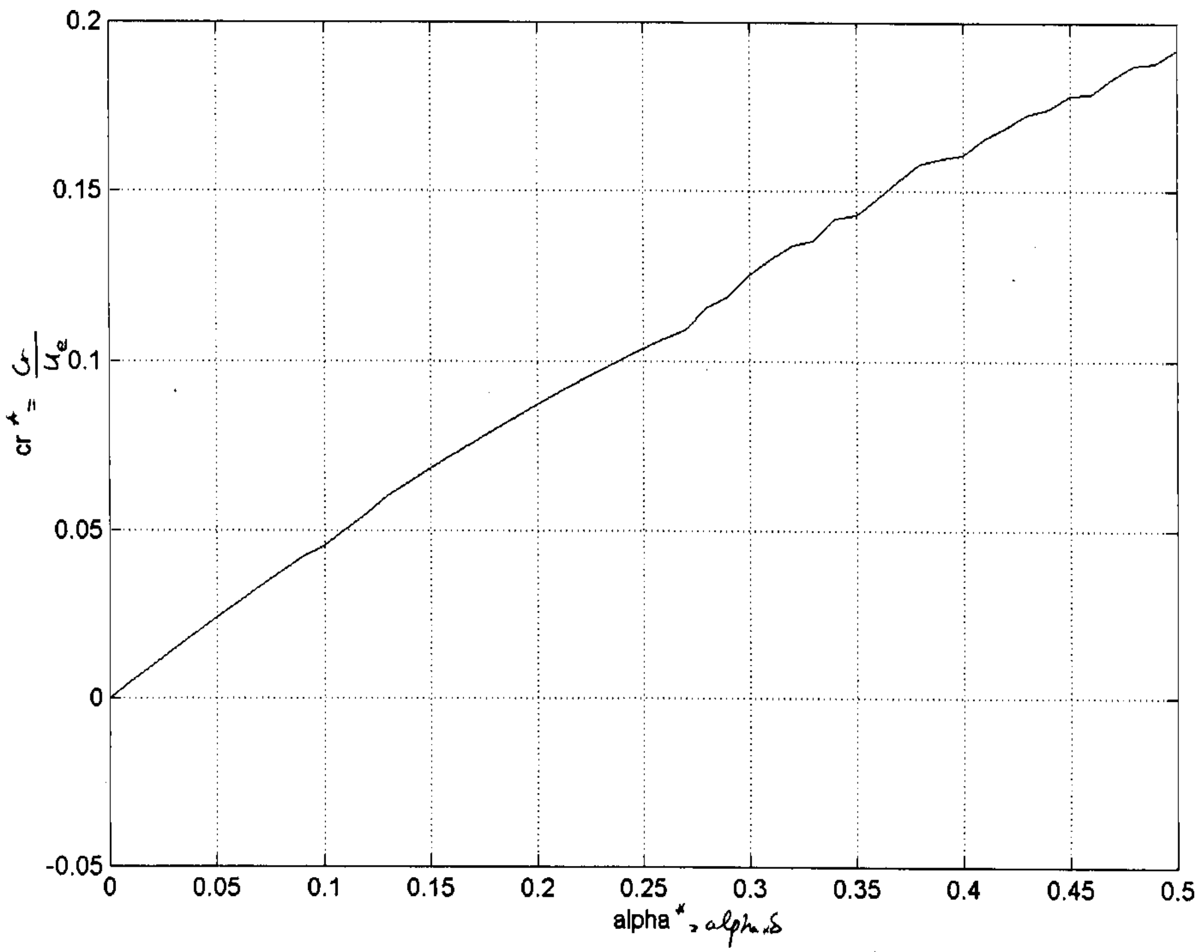
$$\left.\begin{aligned} C_{r_1} &\longrightarrow V_{z_1} \\ C_{r_2} &\longrightarrow V_{z_2} \\ ?\ C_{r_3} &\longrightarrow 0 \end{aligned}\right\} \xrightarrow{\text{linear}} C_{r_3} = C_{r_2} - \frac{C_{r_2} - C_{r_1}}{V_{z_2} - V_{z_1}} V_{z_2}$$

the plots for C_r^* vs $\alpha\delta$ is provided. some eigenfunctions are also plotted.

- If someone wants to solve viscid cases, he should solve a 4^{th} order eigenvalue problem. Now he should guess both C_r and C_i, so there is 2 DOF, but both $v(0) = 0$ & $v'(0) = 0$ should be satisfied. Also both imaginary part and real part should be vanish at $\eta = 0$.



α=0.5
1.4
1.2
1
0.8
0.6
0.4
0.2
0
v(η)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
η



0.2
0.15
0.1
0.05
0
-0.05
cr* = cr/Ue
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
alpha*, alpha.δ

Problem 2:

for $U = kx$, we have:

$$m = 1 \Rightarrow \beta = \frac{2 \times 1}{1+1} = 1$$

$$(Re_{\delta^*})_{crit} = 12,490 \quad (\text{from table 5-1})$$

for $\beta = 1 \Rightarrow \eta^* = 0.6479$

$$\eta = y\sqrt{\frac{(m+1) \times U(x)}{2 \times \nu x}}$$

$$m = 1 \rightarrow \eta = y\sqrt{\frac{U(x)}{\nu x}}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u(x,y)}{U(x)}\right) dy = \int_0^\infty \left(1 - \underbrace{\left(\frac{u(x,y)}{U(x)}\right)}_{f'(\eta)}\right) \frac{d\eta}{\sqrt{\frac{U(x)}{\nu x}}} = \frac{0.6479}{\sqrt{\frac{U(x)}{\nu x}}}$$

$$\Rightarrow \quad \delta^* = \frac{\eta^*}{\sqrt{\frac{U(x)}{\nu x}}} = \frac{0.6479}{\sqrt{\frac{U(x)}{\nu x}}}$$

$$Re_{\delta^*} = \frac{U\delta^*}{\nu} = \frac{U \; 0.6479}{\nu\sqrt{\frac{U}{\nu x}}} = 12,490.$$

$$\sqrt{\frac{Ux}{\nu}} = \frac{12,490}{0.6479} \Rightarrow Re_x = \left(\frac{12490}{0.6479}\right)^2 \Rightarrow \boxed{(Re_x)_{cr} = 3.71 \times 10^8}$$

Problem 3:

Several approaches were possible for this problem. Below is one of the more systematic and accurate ones, based on Twaites correlation method.

This problem is more fascinating than it seems, beleive me!

To find x_i; I use plot 5-12 provided in page 358.

in this graph $Re_{\delta^*, critical}$ is plotted vs $H = \frac{\delta^*}{\theta}$,

to find H, I use "Correlation Method of Thwaites", first we should find λ:

(4-132)

$$\lambda = \frac{\theta^2}{\nu}\frac{du}{dx}$$

$$\theta^2 \approx \frac{0.45\nu}{u^6}\int_0^x u^5 dx = \frac{0.45\nu}{u^6}\int_0^x u_0^5\left(1-\frac{x}{L}\right)^5 dx$$

(4-135)

$$= 0.075\frac{\nu L}{u_0}\left[\left(1-\frac{x}{L}\right)^{-6} - 1\right]$$

$$\Rightarrow \left(\frac{\theta}{L}\right) = \sqrt{\frac{0.075}{\frac{u_0 L}{\nu}}\left[\left(1-\frac{x}{L}\right)^{-6} - 1\right]} \quad \sqrt{\frac{0.075}{10^6}\left[\left(1-\frac{x}{L}\right)^{-6}-1\right]}$$

$$Re_{\delta^*} = \frac{U_0\delta^*}{\nu}\frac{\theta}{\theta}\frac{L}{L} = \frac{u_0 L}{\nu}\frac{\theta}{L}\frac{\delta^*}{\theta} = 10^6\left(\frac{\theta}{L}\right)H \quad (2)$$

$$\lambda = -0.075\left[\left(1-\frac{x}{L}\right)^{-6} - 1\right] \quad (1)$$

So I use the following approach:

first I guess for $\frac{x}{L}$, then using ① I find the λ, then by table 4-4 I find H. after that using (5-12) I can find $Re_{\delta^*,crit}$,
I also can find it by ②, and this 2 should be in agreement, so based on this difference, I update my guess about $\frac{x}{L}$.

first guess:

$$\frac{x}{L} = 0.1 \longrightarrow \lambda = -0.075\left((1-0.1)^{-6} - 1\right) = -0.066126$$

$$H = 3.07$$

$$\frac{\theta}{L} = 2.57 \times 10^{-4}$$

$$5\text{-}12 \longrightarrow Re_{\delta^*} = 10^{2} = 100$$

$$② \longrightarrow Re_{\delta^*} = 10^{6} \times 2.57 \times 10^{-4} \times 3.07 = 789$$

2nd guess:

$$\frac{x}{L} = 0.05 \longrightarrow \lambda = -0.027$$

$$H = 2.73$$

$$\frac{\theta}{L} = 1.64 \times 10^{-4}$$

$$5\text{-}12 \longrightarrow Re_{\delta^*} = 2 \times 10^{2} = 200$$

$$② \longrightarrow Re_{\delta^*} = 448$$

3rd guess

$$\frac{x}{L} = 0.04 \longrightarrow \lambda = -0.0208$$

$$H = 2.69$$

$$\frac{\theta}{L} = 1.44 \times 10^{-4}$$

$$5\text{-}12 \longrightarrow Re_{\delta^*} = 200$$

$$② \longrightarrow Re_{\delta^*} = 388$$

4th guess

$\frac{x}{L} = 0.03 \longrightarrow \lambda = 0.015$

$H = 2.67$

$\frac{\theta}{L} = 1.22 \times 10^{-4}$

$5\text{-}12 \longrightarrow Re_{\delta^*} = 250$

$(2) \longrightarrow Re_{\delta^*} = 325$

5th guess

$\frac{x}{L} = 0.02 \longrightarrow \lambda = 0.00967$

$H = 2.65$

$\frac{\theta}{L} = 9.73 \times 10^{-5}$

$5\text{-}12 \longrightarrow Re_{\delta^*} \approx 260$

$(2) \longrightarrow 260$

$\Rightarrow \boxed{\frac{x}{L} = 0.02}$