

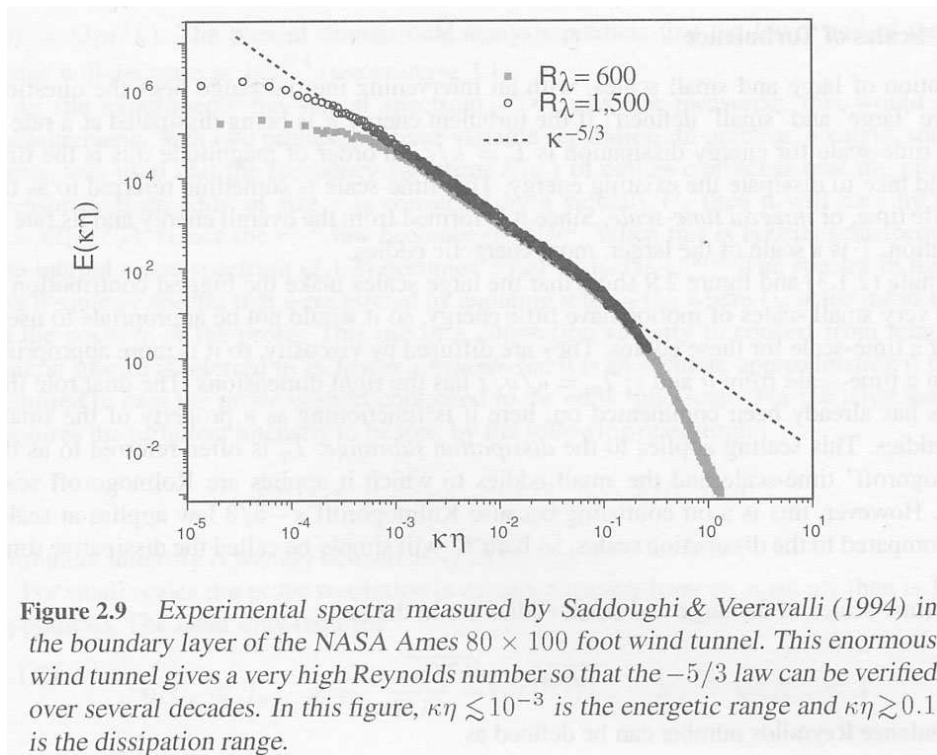
## Answers to Homework #9

### Problem 1:

1. We want to express the kinetic energy per unit wavelength  $E(k)$ , of dimensions  $L^3T^{-2}$ , as a function of the local rate of energy dissipation  $\epsilon$ , of dimensions  $L^2T^{-3}$  (velocity square per unit time) and of the wavenumber  $k$  (dimension  $L^{-1}$ ). To eliminate the time dimension, divide  $E(k)$  by  $\epsilon^{2/3}$ :  $E(k)/\epsilon^{2/3}$  now has dimension  $L^{3-4/3} = L^{5/3}$ . Therefore  $E(k)k^{5/3}/\epsilon^{2/3}$  is a dimensionless parameter. This can be written:

$$E(k) \propto \epsilon^{2/3} k^{-5/3}.$$

This seemingly simple relation, usually called Kolmogorov's ‘ $-5/3$  law’, is an excellent illustration of the power of dimensional analysis. It is verified experimentally, as shown on the figure below. Note that some assumptions are required to justify that  $E(k)$  should only depend on  $\epsilon$  and  $k$  (and not on viscosity or on the length scale of the problem). These assumptions originate from the idea of a turbulent energy cascade: energy is injected into the flow at a macroscopic length scale, and is dissipated at small scales by viscosity; between these two scales there exists an ‘energy cascade’ by which the kinetic energy of the large turbulent eddies is transferred to smaller eddies at a constant dissipation rate  $\epsilon$ . It is in this intermediate range, called inertial subrange, that the  $-5/3$  law is valid.



**Figure 2.9** Experimental spectra measured by Saddoughi & Veeravalli (1994) in the boundary layer of the NASA Ames  $80 \times 100$  foot wind tunnel. This enormous wind tunnel gives a very high Reynolds number so that the  $-5/3$  law can be verified over several decades. In this figure,  $k\eta \lesssim 10^{-3}$  is the energetic range and  $k\eta \gtrsim 0.1$  is the dissipation range.

2. Again, use dimensional analysis. The dimensions of  $\nu_t$  are  $L^2/T$ . It is straightforward to see that from  $\nu_t$ ,  $\Delta$  (length) and  $\epsilon$  the following dimensionless parameter can be constructed:  $\Pi = \nu_t/(\epsilon^{1/3}\Delta^{4/3})$ . Therefore:

$$\nu_t \propto \epsilon^{1/3}\Delta^{4/3}.$$

3.  $E(k)$  is the kinetic energy density per unit wavenumber (and per unit mass). To obtain the total amount of kinetic energy contained in all the filtered out eddies (i.e. in all the wavenumber  $k \geq k_{\max}$ ), integrate  $E(k)$  from  $k_{\max}$  to infinity:

$$K(k \geq k_{\max}) \propto \int_{k_{\max}}^{\infty} E(k)dk = \int_{k_{\max}}^{\infty} \epsilon^{2/3}k^{-5/3}dk = \epsilon^{2/3} \left[ \frac{k^{-2/3}}{-2/3} \right]_{k_{\max}}^{\infty} = \frac{3}{2} \frac{\epsilon^{2/3}}{k_{\max}^{2/3}},$$

and since  $k_{\max} = 2\pi/\Delta$ ,

$$K(\Delta) \propto \frac{3}{2} \frac{\epsilon^{2/3}\Delta^{2/3}}{(2\pi)^{2/3}}.$$

### Problem 2:

1. We use the following values:  $L = 1$  m,  $U = 1$  m/s, and  $\nu = 1.5 \times 10^{-5}$  m<sup>2</sup>/s. For the laminar boundary layer (White, pp. 235):

$$\delta_L \approx \frac{5.0 \times L}{\sqrt{Re_L}} \approx 0.0194 \text{ m}.$$

For the turbulent boundary layer (White, pp. 430):

$$\delta_T \approx \frac{0.16 \times x}{Re_L^{1/7}} \approx 0.033 \text{ m}.$$

In general,  $\delta_L \propto x^{1/2}$  (slow growth, like a parabola), and  $\delta_T \propto x^{6/7}$  (almost linear growth).

2. Transition from laminar to turbulent boundary layer occurs near  $Re_x \approx 5 \times 10^5$ , which in our case corresponds to  $x \approx 3$  m. Unless the boundary layer is tripped, transition will not occur before the end of the plate.
3. The total drag (per unit depth) on the plate is obtained by integrating the wall friction over the length of the plate:

$$D = \frac{1}{2}\rho U^2 \int_{x=0}^{1 \text{ m}} C_f(x)dx.$$

In the laminar case,  $C_f \approx 0.664Re_x^{-1/2}$ , therefore (using  $\rho = 1$  kg/m<sup>3</sup>):

$$D_L = 0.664(1.5 \times 10^{-5})^{1/2} \times \left[ \sqrt{x} \right]_{x=0}^{1 \text{ m}} \approx 2.57 \times 10^{-3} \text{ N/m}.$$

In the turbulent case,  $C_f \approx 0.027Re_x^{-1/7}$ , from which:

$$D_T = \frac{1}{2} \times 0.027(1.5 \times 10^{-5})^{1/7} \times \left[ \frac{x^{6/7}}{6/7} \right]_{x=0}^{1 \text{ m}} \approx 3.22 \times 10^{-3} \text{ N/m}.$$

In general,  $D_L \propto x^{1/2}$ , while  $D_T \propto x^{6/7}$ .

4. The heat transfer equivalent of the wall friction coefficient is the Stanton number, defined as follows:

$$C_h = \frac{q_w}{\rho U c_p (T_w - T_e)}$$

For laminar flow, the Reynolds analogy leads to:


$$C_h^L \approx \frac{C_f}{2Pr^{2/3}},$$

while for turbulent flow, it can be written (cf. White pp. 485):

$$C_h^T \approx \frac{C_f/2}{1 + 13(Pr^{2/3} - 1)(C_f/2)^{1/2}}.$$

These expressions can be used to integrate  $q_w$  over the length of the plate in order to get the precise heat flux across the surface of the plate. To assess the heat transfer capabilities, let's simply evaluate the Stanton numbers in both cases at  $x = 1$  m; using  $Re_L = 67,000$  and  $Pr = 0.7$ :

$$C_h^L \approx 1.63 \times 10^{-3} \quad \text{and} \quad C_h^T \approx 3.23 \times 10^{-3}$$

Therefore, the heat transfer capability of the turbulent boundary layer is twice that of the laminar case. 

5. If the flow velocity is doubled, the boundary layer thicknesses change as follows:

$$\delta'_L = \delta_L/\sqrt{2} \approx 0.0137 \text{ m}, \quad \delta'_T = \delta_T/2^{1/7} \approx 0.0299 \text{ m}.$$

The transition stills occurs at the same Reynolds number. If  $U$  is doubled, this means that it occurs at half the original distance, i.e. around 1.5 m (still no transition on the plate). The drag depends on  $U$  as  $U^{3/2}$  in the laminar case, and  $U^{13/7}$  in the turbulent case, therefore:

$$D'_L = 2^{3/2} D_L \approx 7.27 \times 10^{-3} \text{ N/m}, \quad D'_T = 2^{13/7} D_T \approx 1.17 \times 10^{-2} \text{ N/m}.$$

The dependence of the heat transfer on  $U$  is not as simple for the turbulent case (no simple scaling due to the term involving  $C_f$  in the denominator). The new values of  $C_h$  are  $C_h^L \approx 1.15 \times 10^{-3}$  and  $C_h^T \approx 2.9 \times 10^{-3}$ . Note however that the actual heat transfer is obtained by multiplying  $C_h$  by  $\rho U c_p \Delta T$ , which is itself multiplied by 2 when the flow velocity is doubled.

**Problem 3: (Sangwook Park)**

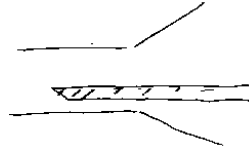
$$\frac{d\theta}{dx} + (2+H) \frac{\theta}{U_e} \frac{dU_e}{dx} = \frac{C_f}{2}$$

\* if there is no pressure gradient,  $\rightarrow \frac{d\theta}{dx} = \frac{C_f}{2}$

Separation  $\Rightarrow$   $C_f = 0$

(adverse pressure gradient?)

$$\frac{d\theta}{dx} + (2+H) \frac{\theta}{U_e} \frac{dU_e}{dx} = 0$$



(continued on the next page)



$$\frac{dU_e}{dz} = - \frac{1}{(2+H)} \frac{U_e}{\theta} \frac{d\theta}{dz}$$

Separation ( $C_f=0$ ) is predicted at  $H=3.0$  (P. 450 in White).

$$\left( \uparrow C_f \approx 0.058 Re_0^{0.268} (0.93 - 1.95 \log_{10} H)^{1.705} \right)$$

$$\therefore \frac{dU_e}{dz} = - \frac{1}{5} \frac{U_e}{\theta} \frac{d\theta}{dz}$$

Separation ( $C_f=0$ ) occurs at  $\xi = \frac{1}{2}$  (P. 953 in White)

$$\Rightarrow \xi = \frac{\delta^*}{\delta} = \frac{1}{2}$$

$$\Rightarrow \delta^* = \frac{1}{2} \delta \Rightarrow H = \frac{\delta^*}{\theta} = 3 \Rightarrow \theta = \frac{1}{3} \delta^* = \frac{1}{6} \delta$$

From Figure 6-29.

$$\Rightarrow \left( \frac{d\theta}{dz} = \text{slope of second figure} \approx \frac{0.08 \text{ cm}}{0.1 \text{ m}} = \frac{0.818}{50} = 0.0016 \right.$$

$$\left. \theta \approx 0.0016z + 0.0003 \right.$$

almost constant.

(It will not change so much)

at separation point

$$\Rightarrow \theta = \frac{1}{6} \delta = 0.16 \times z \times \left( \frac{\nu}{Uz} \right)^{1/4} \times \frac{1}{6} = 0.0016z + 0.0003$$

$$\Rightarrow z \approx 0.1 \text{ m}$$

$$\Rightarrow \theta \approx 0.00046 \text{ m} \Rightarrow 0.046 \text{ cm}$$

$$\Rightarrow \text{using } \theta \approx 0.046 \text{ cm}, U_e = 33 \text{ m/s}, \frac{d\theta}{dz} = 0.0016$$

$$\Rightarrow \frac{dU_e}{dz} \approx -23 \quad \leftarrow \rho \left( U_e \frac{dU_e}{dz} + \nu \frac{d^2 U_e}{dz^2} \right) = -\nabla P.$$

$$\Rightarrow \nabla P = -\rho U_e \frac{dU_e}{dz} \approx 921 \text{ kg/m}^2 \cdot \text{s}^2.$$