Well-Known Boundary Layer Solutions

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Abstract

There is a short list of analytical solutions for the steady, two-dimensional (two spatial variables) laminar boundary layer equations—keeping all the non-linear convective terms.

As I announced earlier, the Mid-Term will be a closed-book/notes take home of finite time, probably two hours. Honor system.

I am not assigning any homework for this Mid-Term week. As you can see in this week's notes, we are just exploiting the dimensional analysis tool this week to study the short list of known analytical solutions. Study the notes. I am likely to find something very similar and ask you to try your hands. You are NOT expected to memorize any detailed formulas—except physical dimensions of things you choose to talk about.

Remember: my emphasis is always physical understanding and insights. And I do not tolerate answers with wrong dimensions.

1 Introduction

In general, we need to deal with PDEs, *partial diffential equations.* For steady, two-dimensional problems, the independent variables are x and y. We are interesting in finding the velocity profile u and the boundary layer thickness δ, in terms of the spatial variables x, y, the reference velocity U_*, the characteristic length L_*, the characteristic density ρ_* and the characteristic viscosity μ_*. There are 8 dimensional variables (2 "output" variables u and δ, and 6 "input" variables and parameters). A little checking shows that in general we can expect 5 dimensionless variables or parameters. A common choice would be:

$$\mathcal{U} \equiv \frac{u}{U_*}, \qquad \text{and} \qquad \frac{\delta}{L_*}, \tag{1}$$

$$\xi \equiv \frac{x}{L_*}, \tag{2}$$

$$\eta \equiv \frac{y}{\delta}, \tag{3}$$

$$R_e \equiv \frac{\rho_* U_* L_*}{\mu_*}. \tag{4}$$

No physics has been involved. Only dimensional analysis (of course, the choices made were insightful choices).

If we use these dimensionless variables in Prandtl's laminar boundary layer equation, we can deduce the additional conclusion that δ/L_* is inversely proportional to $R_e^{1/2}$.[1]

2 Specification of the Problem

The dimensional freestream specification $U_\infty(x/L_*)$ is now rewritten dimensionlessly as $\mathcal{U}_\infty(\xi) = U_\infty(\xi)/U_*$.

Notice: the specification of $U_\infty(x/L_*)$ provides us with <u>both</u> the values of U_* <u>and</u> L_*. When the problem is specified in term of $U_\infty(x/L_*)$, it can be shown that the relations between properly non-dimensionalized variables and parameters do not involve R_e. For such problems the desired solutions are $\mathcal{U}(\xi, \eta)$ and $\delta(\xi)/L_*$.

What happens if the specification of $U_\infty(x/L_*)$ provides only one dimensional parameter? It is intuitively obvious that you would have one less dimensionless parameters. Which of the above dimensionless parameters if dropped would cheer up a fluid mechanicist? The unanimous choice is—without any question—ξ. Why? Because once ξ is out of the game, our PDE problem reduces to an *ordinary differential equation* (ODE) problem. Note that for such problems, the dimensionless parameter analogous to R_e may stay in the ODE problem.

How about (upstream) initial condition? To get analytical solutions, we will take whatever we can get. The means justifies the end.

[1] If we use these dimensionless entities in the Navier-Stokes equation, R_e will stay in the game.

3 Examples

Here comes the short list of known similar solutions.

3.1 Blasius Solution

The zero-pressure gradient case says $U_\infty = U_*$. So no specific L_* is specified. So, why not pick $L_* = x$? Done! Messy around with the equation a bit, we can deduced that

$$\frac{\delta}{x} = C \left(\frac{\rho_* U_* x}{\mu_*} \right)^{-1/2} \tag{5}$$

where C is an $O(1)$ dimensionless number will keep the laminar viscous term happy while it co-exists with the convective terms. To obtain $\mathcal{U}(\eta)$ is a simple matter of solving an ODE by some methods (numerics is totally straightforward).

3.1.1 Two-point boundary value problems

On page 234 of White, the ODE governing the Blasius solution is given by eq.(4.45):

$$f''' + ff'' = 0 \tag{6}$$

where superscript prime denotes differentiation with respect to η. The boundary conditions are:

$$f(0) = 0, \tag{7}$$

$$f'(0) = 0 \tag{8}$$

$$f'(\infty) = 1 \tag{9}$$

This is a so-called "two-point boundary condition" problem—two boundary conditions are given at $\eta = 0$, and one boundary condition is given at the other end. If you use a straightforward ODE solver, you will need to guess at the value of $f''(0)$, do the numerical integration, and then tweak your guess value until your computed $f'(\infty)$ is 1.0. This is affectionately called the "shooting method" in the community.

Here is a popular trick. Do the following transformation:

$$f = \alpha F \tag{10}$$

$$\eta = \frac{Y}{\alpha} \tag{11}$$

and look for $F(Y, \alpha)$ instead of $f(\eta)$ where α is some constant. The ODE for $F(Y)$ is easy obtain:

$$F_{YYY} + F F_{YY} = 0. \tag{12}$$

And the boundary conditions are:

$$\begin{aligned} F(0) &= 0, & (13)\\ F_Y(0) &= 0, & (14)\\ F_Y(\infty) &= \frac{1}{\alpha^2}. & (15) \end{aligned}$$

So, what is the big deal about this transformation?

The big deal is that eq.(12) and eq.(13)-eq.(14) do not depend on α. The relations between $f''(0)$ and $F_{YY}(0)$ and $f'(\infty)$ and $F_Y(\infty)$ are given by:

$$\begin{aligned} f''(0) &= \alpha^3 F_{YY}(0), & (16)\\ f'(\infty) &= \alpha^2 F_Y(\infty), & (17) \end{aligned}$$

which are indeed α dependent. So you can pick any (positive) value for $F_{YY}(0)$ you like (how about unity?) and use eq.(16) and eq.(13)-eq.(14) to numerically integrate eq.(12) (start from the wall). After the solution is obtain, you will have a computed value for $F_Y(\infty)$ which is of course not unity (but presumed positive). You can now find out from eq.(17) what α can make $f'(\infty)$ precisely unity, as demanded by eq.(9). Your original problem for $f(\eta)$ is thus solved—without the need for the shooting method. [2]

3.2 The Falkner-Skan Family

The Falkner-Skan Family considers:

$$U_\infty = K x^m. \tag{18}$$

What is the dimension of K? It is U_*/L_*^m. We have no U_*, nor L_*; we just have K.

$$\frac{\delta}{x} = C\left(\frac{\rho_* U_\infty x}{\mu_*}\right)^{-1/2} = C\left(\frac{\rho_* K x^{m+1}}{\mu_*}\right)^{-1/2} \tag{19}$$

[2]Note: this trick works only if one boundary condition is on the other side, and the ODE and the remaining bondary conditions of the transormed system are independent of the transformation parameters.

or

$$\delta = C\ x \left(\frac{\rho_* U_\infty x}{\mu_*} \right)^{-1/2} = C \left(\frac{\mu_* x^{1-m}}{\rho_* K} \right)^{1/2} \tag{20}$$

The final ODE (White's eq.(4-71)) depends on m but not on K.

It turns out that $m = 0$ and $m = 1$ have obvious interpretations, while other values of m can be associated with flows over two dimensional wedges—except for the case $m = -1$. The dimensionless parameter β (defined by eq.(4-71)) is often preferred over m. See Fig. 4-10 on page 243 of White for physical meaning of β.

Note that m is actually $d \ln U_\infty / d \ln x$. So m or $\beta > 0$ means the pressure gradient is favorable, while m or $\beta < 0$ means it is unfavorable. Look at Fig. 4-11a on page 245 of White. Note that positive β's are boundary layers with favorable pressure gradients, while negative *beta*'s are for unfavorable pressure gradients. Note that the velocity profiles have an "inflection point" only when m or $\beta < 0$.

3.3 The $m = -1$ case

What happens to White's eq.(4.71) when $m \to -1$? Do you really need to go to White's §4.3-7, or can you persevere on here with the Falkner-Skan family?

When $m \to -1$, β blows up. Now, whenever something blows up, always go there by following a "limit" protocol. Consider the transformation:

$$f = \alpha F \tag{21}$$

$$\eta = \frac{Y}{\gamma} \tag{22}$$

where again α and γ are arbitrary positive numbers. Now, the question is simply: what should you pick for α and γ so that your original ODE (i.e. eq.(4-71)) keeps the viscous term competitive in the $\beta \to \infty$ limit? Voila, you can recover eq.(4-88) all by yourself!

3.4 The Plane Laminar Jet

Suppose instead of U_∞, you are given the total x-momentum flux of a two-dimensional steady laminar jet issuing from the origin toward the the east into an otherwise quiescent fluid domain:

$$J = \rho_* \int_{-\infty}^{+\infty} u^2 dy \tag{23}$$

If you mess around a bit, you will find the following are dimensionally correct dimensional velocity and length:

$$U_* = \frac{J^{2/3}}{(\rho_* \mu_* x)^{1/3}}, \quad (24)$$

$$\delta = \left(\frac{\mu_*^2 x^2}{\rho_* J} \right)^{1/3}. \quad (25)$$

The dimensionless $\mathcal{U}$ is a function of η and not of ξ, and not of R_e.[3] Assuming that a solution exists, we see that the jet velocity (e.g. centerline) is proportional to $x^{-1/3}$ and the width of the jet is proportional to $x^{2/3}$. And you can figure out easily what happens if J is increased 8 fold.

Remember, for this kind of problem specification, the dimensionless parameter analogous to R_e may stay in the game. For the two-dimensional case, this parameter did not get involved. You can try your hands on extending the analysis to the axisymmetric laminar jet, and compare you "answers" for U_* and δ to the narrow axi-symmetric jet studied on page 301, §4-10.6 of White. You will find that a simple-minded extension (changing the dimension of J) *not* could come up with the right answers. The dimensionless C defined by eq.(4-206) is essentially the parameter analogous to R_e which now participates in the answer. So: dimensional analysis alone is not enough. Must get the equations involved to be certain.

3.5 The Plane Laminar Wake

In White's §4.4-3, he considered the far-field of a plane, steady two-dimensional laminar wake behind an obstacle that experienced a drag F (force per unit length) in a uniform flow with velocity U_o.

By limiting his attention to the far-field, he expect u to approach U_*, so the velocity "defect" $u_1 \equiv U_o - u$, is small. It is clear that this defect must be zero if F is zero. So it is reasonable to expect the defect to be proportional to F.

[3]Yes, I cheated—by peeking at the answers. Remember: dimensional analysis is not physics. Whether the choice is the "good" choice must be verified by mathematics using the equations (or by experiments). A good choice is one which allows the dependence on one or more dimensionless parameters to be ignored. Once the choices are verified to be the right choices, they talks to you loud and clear.

With this hint, it is possible to get a similar solution for this problem—instead of solving (using the separation of variable trick) the linear PDE as exhibited in eq.(4-109). Pick:

$$U_* = \frac{F}{\rho_* U_o \delta} \tag{26}$$

$$\delta = \left(\frac{\mu_*}{\rho_* U_o}\right)^{1/2} \tag{27}$$

Again, your dimensionless $\mathcal{U}_1 = u_1/U_*$ is a function of η only. Note δ is proportional to $x^{1/2}$.

4 How thick are thermal (temperature) boundary layers?

The Prandtl Number P_r, is defined as:

$$P_r = \frac{\mu C_p}{\kappa} \tag{28}$$

where C_p is the specific heat at constant pressure. It is dimensionless, and is indicative of the impact of viscosity versus heat conductivity.

For most gases, P_r is somewhat less than unity. For liquids, the range is much wider.

Hence, roughly speaking, a thermal boundary layer is approximately of the same order of magnitude as the viscous boundary layer—for gases.

With this hint, you are now also capable of making good guesses for heat transfer problems (for gases).

5 *ad hoc* Methods

There are many *ad hoc* methods, developed before the digital computer became a household appliance.

The momentum integral equation is one of the best known. Make sure you can do the derivation (add $(u - U_\infty)(\nabla \cdot \mathbf{V}) = 0$ to the x-momentum boundary layer equation, then mess around). Learn the general idea of the Thwaites Method.

6 Numerical Marching Algorithms

But now all fluid mechanics students have computers. We will take a crack at the Prandtl's steady two-dimensional laminar boundary layer equations next week. Brush up your Matlab.

Look at §4-5.1, the linearly retarded flow of Howarth. To obtain an eight term Taylor Series (see eq.(4-115) is a lot of messy uninteresting work. It is of historical interest to know that Howarth (1938) has done just a few terms, and Professor Van Dyke used computer (1984) and very clever numerical tricks to get the rest of the displayed terms. With Matlab, you can do *all* (steady, two-dimensional, laminar) boundary layer problems with one tiny Matlab program.