

Low Reynolds Number Flows

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Abstract

The Reynolds Number R_e is the most important dimensionless parameter in viscous flow. Here we study the Low Reynolds Number regime. How low? Asymptotically low. We are interested in tiny things moving slowly in viscous stuff. How tiny, how slowly, and how viscous? Well, its Reynolds Number is very small.

1 Introduction

We shall assume density and viscosity to be constants.

In the limit of vanishingly small Reynolds Number R_e , the inertial term (normally on the left side of the momentum equation) can be neglected. The resulting momentum equation is attributed to Stokes, and flows described by this equation are called Stokes Flows.

The governing equations are then:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$-\nabla p + \mu \nabla^2 \mathbf{V} = O(R_e). \tag{2}$$

We have a scalar equation and a vector equation for a scalar unknown p and a vector unknown \mathbf{V} . Note that the right hand side of eq.(1) is precisely zero, while the right hand side of eq.(2) is something small that we are neglecting in the small R_e limit.

1.1 The issue of memory

Note that the $\partial\mathbf{V}/\partial t$ term has been neglected. Thus, once a solution $\mathbf{V} = \mathbf{S}(x, y, z, A(t))$, $p = P(x, y, z, B(t))$ have been found for the Stokes Flow equations for constant A and B , this solution is also valid for any $A(t)$ and $B(t)$. In other words, the flow field is dependent only on the current value of $A(t)$ and $B(t)$. Thus, Stokes Flows have no memory.

1.2 Dimensional analysis

What do you want to know? Before we dive into finding solutions, lets see how far we can go with dimensional analysis.

1.2.1 Pressure Drop in a Long Pipe

We want to know the amount of pressure drop Δp to push a fluid with density ρ and viscosity μ through a pipe with diameter D over a long distance L with centerline flow velocity U .

Counting the above, we have six dimensional parameters. Checking the rank of the relevant matrix,¹ we confirm that the rank is three. So we have $6 - 3 =$ three dimensionless parameters.

Here is a set of intelligent (non-unique) choices:

$$\Pi_1 = \frac{\Delta p D^2}{\mu U L}, \quad \Pi_2 = \frac{\rho U D}{\mu}, \quad \Pi_3 = \frac{L}{D}. \quad (3)$$

So, our goal is to find

$$\Pi_1 = \mathcal{F}(\Pi_2, \Pi_3). \quad (4)$$

In general, we are interested in very large Π_3 's. For Stokes Flows, we are interested in very small Π_2 's. In other words, we are interested in $\mathcal{F}(\Pi_2 \rightarrow 0, \Pi_3 \rightarrow \infty)$ —if it exists.²

Now, what happens if we are interested in very large Π_2 's (large Reynolds Numbers)? Dimensional analysis must remain applicable. Following the previous rationale, we would be looking for $\mathcal{F}(\Pi_2 \rightarrow \infty, \Pi_3 \rightarrow \infty)$ —if it exists.

¹I assume you know what I am talking about.

²Actually, for the long pipe (*Poiseuille* flow) problem, the inertia term can be neglected based on the $\Pi_3 \rightarrow \infty$ approximation, while Π_1 is allowed to have any value at all. Physically, if Π_3 is long enough, the fluid particles must experience negligible acceleration in the streamwise direction as it flows downstream.

It turns out that for this case, Π_1 is expected to be proportional to Π_2 for large Π_2 (justified by experimental observations). A more intelligent choice would be to replace Π_1 by Π_4 , defined by

$$\Pi_4 = \frac{\Pi_1}{\Pi_2} = \frac{\Delta p D}{\rho U^2 L}. \quad (5)$$

The goal is now to find

$$\Pi_4 = \mathcal{G}(\Pi_2, \Pi_3 \rightarrow \infty). \quad (6)$$

Of course, this is essentially the Moody diagram—except that Moody told us that we should have been included the roughness of the pipe in our original list of relevant dimensional parameters. Most importantly, the diagram puts a spotlight on the phenomena of laminar to turbulent transition for all to see.

1.2.2 Drag of a sphere

We want to know the drag \mathcal{D} of a sphere of radius R in a steady flow at velocity U in a fluid of density ρ and viscosity μ . Here we have five dimensionless parameters, and a little checking tells us that we can expect two dimensionless parameters.

Here is a good (non-unique) pick:

$$\Pi_1 = \frac{\mathcal{D}}{\mu U R} \quad \Pi_2 = \frac{\rho U R}{\mu}. \quad (7)$$

Here is another good (non-unique) pick:

$$\Pi_1 = \frac{\mathcal{D}}{\rho U^2 R^2} \quad \Pi_2 = \frac{\rho U R}{\mu}. \quad (8)$$

Which one would you pick for Stokes Flow problems? Which one would you pick for high Reynolds Number problems? Remember, the issue is not which choice is correct. The issue is how smart (and insightful) you are.

Totally throwing out the inertia term is called the “Stokes approximation.” If it is desired to find a “correction” when Π_2 is small but finite, one may use a Taylor Series expansion for \mathcal{F} about $\Pi_2 = 0$. When the series keeps only two terms, the approximation is at the level of the so-called “Oseen approximation.” (see later).

1.2.3 The viscous length

Suppose I have a sphere of diameter D in a fluid flow. How big is the region of the flow field that feels the presence of the sphere?

For high Reynolds Number flows, the characteristic size of the region being affected is $O(D)$. What happens when D is very, very small so that the Reynolds Number is very, very small? Does the (dimensional) size of the region being affected continue to be proportional to D ?

Dimensional analysis suggests an interesting dimensional length that is independent of D —the length that would make the Reynolds Number unity. So, one speculation is that as $D \rightarrow 0$, the size of the disturbed region approaches $\mu/(\rho U)$ in the limit. This is pure speculation. Whether it is right or not can be decided only by rigorous theories and/or observations.

1.2.4 The Lubrication Problem

The classical lubrication problem is shown by Fig. 3.41 on page 187 of White. The dimensionless parameters are: Δp , U , ρ , μ , L , h_o , h_L , x and y . So we can expect six dimensionless parameters.

Here is a good pick:

$$\Pi_1 = \frac{\Delta p h_o^2}{\mu U L}, \quad \Pi_2 = \frac{\rho U h_o}{\mu}, \quad \Pi_3 = \frac{h_o}{h_L}, \quad \Pi_4 = \frac{x}{L}, \quad (9)$$

$$\Pi_5 = \frac{h_o}{L}, \quad \Pi_6 = \frac{y}{h_o}. \quad (10)$$

So our goal is to look for:

$$\Pi_1 = \mathcal{H}(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6). \quad (11)$$

For Stokes Flows, we are interested in the $\Pi_2 \rightarrow 0$ regime. In addition, we are interested in problems with Π_5 very small—thus we can set $\Pi_5 \rightarrow 0$. So the major contribution of detailed mathematical activities is to obtain the dependence of \mathcal{H} on Π_3 and Π_4 . See White's eq.(3-240) on page 188. Note that this solution remains valid if U , L , h_o and h_L are time dependent, and that the solution has no memory.

2 Doing the lubrication problem more methodically

We now consider the limiting case of Π_2 and Π_5 being both very small.

2.1 Solving for pressure

If we take the divergence of eq.(2) and take advantage of eq.(1), we have

$$\nabla^2 p = O(R_e). \quad (12)$$

Let us now introduce dimensional coordinates:

$$\xi = \frac{x}{L} = \Pi_4, \quad \eta = \frac{y}{h_o} = \Pi_6. \quad (13)$$

We can rewrite eq.(12) as:

$$\frac{\partial^2 p}{\partial \eta^2} + \Pi_5^2 \frac{\partial^2 p}{\partial \xi^2} = O(\Pi_2). \quad (14)$$

In the limit of small Π_2 and Π_5 , we have:³

$$\Pi_1 = \mathcal{P}(\xi) + O(\Pi_2, \Pi_5). \quad (15)$$

We define \mathcal{U} by:

$$\mathcal{U} = \frac{u}{U}. \quad (16)$$

Substituting into the x -component of eq.(2), we obtain:

$$-\frac{d\mathcal{P}}{d\xi} + \frac{\partial^2 \mathcal{U}}{\partial \eta^2} = O(\Pi_5^2, \Pi_2). \quad (17)$$

Integrating this equation twice and imposing the no-slip condition on the upper and lower (moving) walls, we get the velocity profile $\mathcal{U}(\xi, \eta)$ with $d\mathcal{P}/d\xi$ as a “parameter.” See eq.(3-237) on page 188 of White. Imposing the condition that the (yet unknown) mass flux in the x -direction is independent of ξ , we get a first order ordinary differential equation for $\mathcal{P}(\xi)$. The solution is readily obtained—it can satisfy *two* boundary conditions at the entrance and the exit of the narrow gap—thereby determining the previously unknown mass flux in the process. The “error” of this solution is $O(\Pi_2, \Pi_5^2)$.

³Can you justify to yourself why \mathcal{P} can not depend on η linearly in this limit?

Want to improve this solution? This can be done by the so-called “regular” perturbation methodology—it is just very tedious work. The strategy is very simple: use the current approximate solution to evaluate all the neglected terms, and redo the whole problem again including these terms as known functions. The strategy of *iteration* is most powerful!

3 The Oseen Approximation

Suppose we have this simple equation for X :

$$X + A + B = 0 \tag{18}$$

and we know B is very small (e.g. 1.234×10^{-4}). What would be a “good” approximate solution for X ?

A tentative answer is: $X \approx -A$. Let’s see how good this is when $B = \exp(-r/R)$ where R is some positive constant. For $r/R < 9$, the approximation is pretty good. For $r/R \sim 9$, the approximation is bad. For $r/R > 9$, the approximation is terrible. For the last region, a better approximation is $X \approx -B$.

This simple example tells us to be more careful in using the Stokes approximation. In the original momentum equation, we have three terms: the inertial term, the pressure term, and the viscous term. In the Low Reynolds Number regime, we estimate the inertial term to be small. Does that allow us to neglect it? Not quite so fast! It is negligible only if we can prove that one of the remaining terms (either pressure or viscous) is not small.

For the flow over a solid body of characteristic length R , the pressure and the viscous terms near the solid body are certainly not small (compared to what?). But what happens when you go further out from the body? Both the pressure term and the viscous term are expected to decay toward zero. So the Stokes approximation may fail far away from the body, where a better approximation may be to neglect the viscous term—and let the pressure term and the inertia term balance each other.

The Stokes solution for the flow over a sphere has some features that are not nice. For example, the flow field is symmetric fore and aft. There is no wake. And the Stokes solution for flows over a cylinder has other irritating problems.

So Mr. Oseen proposed an *ad hoc* representation of the inertial term in the momentum equation. He replaced Du/Dt by:

$$\frac{Du}{Dt} \rightarrow U_o \frac{\partial u}{\partial x} \quad (19)$$

where U_o is the upstream flow velocity. This is very bad near the sphere, but there the inertial term is small compared to the pressure and viscous terms there. It becomes better and better far, far away from the sphere, and is then available to be competitive with the pressure term. It gives nicer looking flow fields (it distinguishes upstream from downstream), provides a small correction to the Stokes drag, and provides resolutions to many mathematical irritations.

Most importantly, the Stokes and Oseen Approximations render the governing PDEs *linear*, and allow fairly standard mathematical tools to tackle them.

4 The Knudsen Number

The Knudsen Number K_n is defined as:

$$K_n \equiv \frac{\ell}{L} \quad (20)$$

where ℓ is the characteristic mean-free-path of the gas and L is the characteristic length of the problem.

The *continuum* formulation of fluid mechanics is formally a theory valid in the small K_n limit. In other words, when K_n is not “negligibly small,” the continuum theory needs “corrections.”

It is readily shown (using $\mu \sim \rho a \ell$ where ‘a’ is speed of sound) that

$$K_n = O\left(\frac{M}{R_e}\right) \quad (21)$$

where M is characteristic Mach Number and R_e is characteristic Reynolds Number. Hence, for a low Reynold Number gas flow problems, the magnitude of the Knudsen Number increases as the Reynolds Number decreases—when the Mach Number is held fixed. The question now is: if K_n is not small enough to make the continuum formulation legitimate, what needs to be fixed?

This is a question that needs the intervention of kinetic theory. The short answer is: the Navier-Stokes momentum equation needs some fixing up (this is hard), and the no-slip condition for the fluid-solid interface needs attention (this is relatively painless).

Homework 1: You are trying to slide a heavy weight of mass m (kilograms) with a smooth rectangular flat bottom (of known area $A = L \times W$) on a smooth surface at velocity U . You put some cheap lubricants with viscosity μ_1 on. It is not good enough. Now Sears sells a more expensive lubricant with μ_2 half the value of μ_1 . Estimate (I don't want precise answers; just estimates) the benefit of using the more expensive lubricant. State clearly your assumptions.

Homework 2: Consider the heat transfer problem of a hot finite body with characteristic length D in a cold, low Mach Number (incompressible constant viscosity) flow. Do the dimensional analysis and suggest how experimental data ought to be plotted. Look up for the definition of the Nusselt Number.

Homework 3: Consider the problem of a hot cylinder. (a) What would be the governing PDE for temperature T under the Stokes approximation (in limit of asymptotically small Peclet Number)? (b) Find the analytical solution for the asymptotically small Peclet Number case when the cylinder temperature is held at T_w and the fluid temperature at infinity is held at T_∞ . (This is a tricky question!).⁴ What does your troubles tell you?

Homework 4: Do #3 for the sphere. Find the Nusselt Number under the Stokes approximation (This is very easy).

Homework 5: Look at the lubrication solution obtained by White. What do you need to do if $U(t)$ is time dependent? How "rapid" does the time dependence need to be so that a "correction" to the solution is needed?

⁴Hint: First impose your "infinity" boundary condition for temperature not at infinity, but at $r = K$, get the analytical solution, then take K to the infinity limit.