

ME 451C
Winter Quarter, 2004-05
Final Exam

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Abstract

This open-book/notes final exam, along with a folder of all of this quarter's homeworks (including those you had handed in¹, are due by noon on Friday, March 18th, 2005. Please hand them in to Ms. Deb Michael, Professor Moin's secretary in Building 500.

Watch the course website for instructions on how to get your papers back.

Thank you all for attending.

1 Another toy problem

Consider the $N = 2$ reactive-diffusive PDE for $X_1(y, T)$ and $X_2(y, t)$:

$$\frac{\partial X_1}{\partial t} = W_1 + W_2, \quad (1)$$

$$\frac{\partial X_2}{\partial t} = -\gamma W_1 + W_3, \quad (2)$$

where

$$W_1 = \frac{X_2^2 - X_1^2}{\epsilon}, \quad (3)$$

¹It's OK to edit and make corrections on them

$$W_2 = w_2(X_1, X_2) + D_{11} \frac{\partial^2 X_1}{\partial y^2} + D_{12} \frac{\partial^2 X_2}{\partial y^2}, \quad (4)$$

$$W_3 = w_3(X_1, X_2) + D_{21} \frac{\partial^2 X_1}{\partial y^2} + D_{22} \frac{\partial^2 X_2}{\partial y^2}. \quad (5)$$

Everything had been “intelligently” non-dimensionalized. The parameter γ may be a (differentiable) function of X_1 and X_2 . The domain of interest is $0 \leq y \leq 1$ and $0 \leq t \leq 10$, and ϵ is some small positive number, most probably less than 0.01, but nobody knows the precise value (the error bar on ϵ is big, but it is quite certain that it is a small number). The chemistry terms are:

$$w_3 = 1 \quad (6)$$

$$w_2 = -2X_1^3. \quad (7)$$

The initial conditions are:

$$X_1(y, t = 0) = c_1(y), \quad (8)$$

$$X_2(y, t = 0) = c_2(y), \quad (9)$$

and $c_1(y) \neq c_2(y)$ and are $O(1)$ entities. The boundary conditions are:

$$X_1(y = 0, t) = b_1(t), \quad (10)$$

$$X_2(y = 0, t) = b_2(t), \quad (11)$$

$$\left(D_{11} \frac{\partial X_1}{\partial y} + D_{12} \frac{\partial X_2}{\partial y} \right) (y = 1, t) = 0, \quad (12)$$

$$\left(D_{21} \frac{\partial X_1}{\partial y} + D_{22} \frac{\partial X_2}{\partial y} \right) (y = 1, t) = 0, \quad (13)$$

and $b_1(t)$ and $b_2(t)$ are $O(1)$ entities.

The diffusion coefficients are constants:

$$D_{11} = 2D_{22} = D_{12} = D_{21} = \alpha. \quad (14)$$

where α is some $O(1)$ positive number.

1. Analytically derive the leading order reduced chemistry PDE using the conventional PE approximation—keeping in mind that $\gamma(X_1, X_2)$ is differentiable, $O(1)$ and independent of ϵ .

2. Analytically derive the once-refined (both steps) fast basis vector \mathbf{a}_1^o and \mathbf{b}_o^1 —under the assumption the $W_1 = O(1)$ in the small ϵ limit and γ is not a constant—starting with any choice for the initial guess. Show the fast and slow subspace projection matrices (under the $W_1 = O(1)$ assumption). Derive the CSP reduced chemistry PDEs (valid after the fast mode died) and compare the answer with the answer from (1).
3. For $\gamma = \gamma(X_1, X_2)$ not being a constant, there is no approximate conserved scalar in general. Assume $\gamma = 1 + X_1 X_2$. Numerically compute the initial conditions for the reduced chemistry PDE for $c_1(y) = 1$, $c_2(y) = 0$.
4. Numerically compute the reduced chemistry ODE solution for the pure chemistry case for $\alpha = 0$, $\gamma = 1 + X_1 X_2$, $c_1(y) = 1$, $c_2(y) = 0$. We are interested for $0 \leq t \leq 10$.
5. Analytically derived the governing equations in the thin layer adjacent to $y = 0$ when $b_1(t) \neq b_2(t)$, and provide the appropriate boundary conditions (particularly at the “edge” of this thin layer). You can use classical asymptotic analysis, or CSP, or any good mathematics. Again, assumed $\gamma = \gamma(X_1, X_2)$ so that there is no approximate conserved scalars. You don’t need to do any computation, but need to provide a correctly posed mathematical problem for this layer.
6. We are interested in the solution for the chemistry-diffusion coupled (reduced chemistry— $\epsilon \ll 1$) PDE. Use $\alpha = 1$, $\gamma = 2$, $c_1(y) = 1$, $c_2(y) = 0$ and $b_1(t) = b_2(t) = 0.1$. The time range of interest is $0 < t < 10$.

From what you have learn from the above studies, provide some sketches to show what the reduced chemistry solutions should look like qualitatively.