

ME 451C
Answers to Mid-Term
Winter Quarter, 2005

Harvey Lam

March 9, 2005

Abstract

These notes provide supplementary comments on some of the questions in the mid-term.

1 Mid-Term

A Use the “elementary derivation” to derive the diffusion laws. State all your assumptions. Keep both \mathbf{E} (electric field) and \mathbf{B} (magnetic field) in the game. We agree that in the presence of \mathbf{E} and \mathbf{B} , a charge particle with charge e moving with velocity \mathbf{V} experiences a *Lorentz force* \mathbf{F} given by:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{V} \times \mathbf{B}). \quad (1)$$

For a singly charged negative ion, $q = -e$, and for a singly charged positive ion, $q = e$.

====

We just need to work on the “body force” term. Remember, body force \mathbf{h}_α is force per unit mass of the medium. $\rho_\alpha \mathbf{h}_\alpha$ is force per unit volume. Electromagnetic forces are force per unit charge in the medium.

We shall use subscript $+$ to denote (singly charged) positive ions, and subscript $-$ to denote singly charged negative ions (which may be electrons). The charge of an electron (magnitude) is denoted by e .

We shall assume all positive ions to have the same mass m_+ and all negative ions to have the same mass m_- . We denote number densities of the ions by n_+ and n_- , the mass fractions by Y_+ and Y_- . So we have:

$$n_+m_+ = \rho_+, \quad (2)$$

$$n_-m_- = \rho_-. \quad (3)$$

Note that under the assumption that all positive ions have identical mass and all negative ions have identical mass, mass averaged and number averaged velocities are the same.

So, the “body force per unit volume” for the positive and negative ions in their respective momentum equations are:

$$\rho_+\mathbf{h}_+ = \rho Y_+\mathbf{h}_+ = +en_+(\mathbf{E} + \mathbf{V}_+ \times \mathbf{B}) = +\frac{e\rho_+}{m_+}(\mathbf{E} + \mathbf{V}_+ \times \mathbf{B}), \quad (4)$$

$$\rho_-\mathbf{h}_- = \rho Y_-\mathbf{h}_- = -en_-(\mathbf{E} + \mathbf{V}_- \times \mathbf{B}) = -\frac{e\rho_-}{m_-}(\mathbf{E} + \mathbf{V}_- \times \mathbf{B}). \quad (5)$$

Later, we shall need the sum of these two terms. Adding, we have:

$$\rho(Y_+\mathbf{h}_+ + Y_-\mathbf{h}_-) = e(n_+ - n_-)\mathbf{E} + \mathbf{J} \times \mathbf{B}, \quad (6)$$

where the electric current \mathbf{J} is, by definition, given by:

$$\mathbf{J} \equiv en_+\mathbf{V}_+ - en_-\mathbf{V}_- = e(n_+ - n_-)\mathbf{V} + e(n_+\mathbf{v}_+^* - n_-\mathbf{v}_-^*). \quad (7)$$

We shall need Eq.(6) when we look for the new electrical contributions to the diffusion velocities \mathbf{v}_+^* and \mathbf{v}_-^* .

It is important to note that electric current is mainly carried by the diffusion number fluxes of the charged particles. The contribution by the “convection flux” under the quasi-neutral approximation—which is usually a good approximation away from sheaths—is negligible. Under the quasi-neutral approximation, Eq.(6) becomes:

$$\rho(Y_+\mathbf{h}_+ + Y_-\mathbf{h}_-) \approx \mathbf{J} \times \mathbf{B}. \quad (8)$$

The contribution by electrical forces to the mass diffusion velocity \mathbf{v}_+^* is given by eq.(41) of Week #4 Notes (using Eq.(6) and a diagonal $D_{\alpha\beta}$

under the assumption that the ionization level is low and collisions are mostly with the neutrals):

$$\begin{aligned} \mathbf{v}_+^* (\text{from } \mathbf{H}_+) &= -D_{++} \mathbf{H}_+ = -D_{++} \frac{\rho Y_+}{p} [(Y_+ \mathbf{h}_+ + Y_- \mathbf{h}_-) - \mathbf{h}_+] \quad (9) \\ &= -\frac{D_{++}}{p} \{Y_+ [e(n_+ - n_-) \mathbf{E} + \mathbf{J} \times \mathbf{B}] - en_+ [\mathbf{E} + \mathbf{V}_+ \times \mathbf{B}]\} \end{aligned}$$

We see we still have \mathbf{V}_+ on the right hand side. We can now make the $\mathbf{V}_+ = \mathbf{V} + \mathbf{v}_+^* \approx \mathbf{V}$ approximation¹ to yield:

$$\mathbf{v}_+^* (\text{from } \mathbf{H}_+) \approx -\frac{D_{++}}{p} [Y_+ \mathbf{J} \times \mathbf{B} + e(Y_+(n_+ - n_-) \mathbf{E} - n_+ (\mathbf{E} + \mathbf{V} \times \mathbf{B}))] \quad (10)$$

Similarly, we have:

$$\mathbf{v}_-^* (\text{from } \mathbf{H}_-) \approx -\frac{D_{--}}{p} [Y_- \mathbf{J} \times \mathbf{B} + e(Y_-(n_+ - n_-) \mathbf{E} + n_- (\mathbf{E} + \mathbf{V} \times \mathbf{B}))]. \quad (11)$$

These are the contributions to diffusion velocities arising in response to electrical forces.

B We will ignore Soret and Dufour effects. Compare what you derived with the phenomenological formulas used in class. Identify the conditions under which the phenomenological formulas can be considered a good approximation to your results.

=====

The weakly ionized approximation is needed to justify the matrix $D_{\alpha\beta}$ being diagonal. The low Mach Number approximation is needed to ignore the gradient of the total pressure term.

C Show that you can “derive” the Einstein relation (between diffusion and mobility coefficients) attributed to Einstein. Obviously, you need to state the assumptions needed.

=====

¹This approximation is actually quite difficult to justify when \mathbf{B} is not small. The more justifiable procedure is to keep the \mathbf{v}_+^* term on both sides of the equation, and solve the resulting vector equation for \mathbf{v}_+^* algebraically with considerable amount of manipulations.

The diffusion is driven by pressure gradient ∇p_+ and electrical force $n_+ \mathbf{E}$. From the elementary derivation, the two terms are additive (they are terms in the +momentum equation):

$$(\nabla p_+ + en_+ \mathbf{E}) \tag{12}$$

If we adopt the perfect gas law $p_+ = n_+ kT$ and assume temperature T to be a constant, we have:

$$kT \nabla n_+ + en_+ \mathbf{E} = kT (\nabla n_+ + \frac{e}{kT} n_+ \mathbf{E}) \tag{13}$$

The Einstein relation between diffusion and mobility coefficients is immediately obtained by inspection of this equation.

D Derive the momentum equation for the mixture (valid for arbitrary reference Debye length).

=====

This can be done without going through the “elementary derivation” of the diffusion laws. If you know the electrical forces on charge particles, you can work out the electrical forces on the mixture under the continuum approximation. And the $\mathbf{J} \times \mathbf{B}$ (body force per unit volume) term should show up.

E We confine our attention to problems for which the reference length L_R is much larger than the reference Debye length. Derive the Ohm’s Law under the quasi-neutral approximation, keeping \mathbf{B} in the game.

=====

Again, as we did previously, we split the electric field into three contributions:

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \mathbf{E}_{ambi} + \mathbf{E}_{ohmic} \tag{14}$$

where the first term accounts for the $\mathbf{V} \times \mathbf{B}$ electric field, \mathbf{E}_{ambi} accounts for the ambi-polar electric field (defined the same way as was done previously), and the remaining \mathbf{E}_{ohmic} is the electric field associated with the electric current \mathbf{J} —same as before. The relation between \mathbf{J} and \mathbf{E} is the desired Ohm’s Law.

We shall adopt the quasi-neutral approximation, and denote $n_+ = n_- = n_*$. We now have (using Eq.(10) and Eq.(11)):

$$\mathbf{J} = en_*(\mathbf{v}_+^* - \mathbf{v}_-^*) = \sigma \mathbf{E}_{ohmic} + \gamma \mathbf{J} \times \mathbf{B} \tag{15}$$

where

$$\sigma = \frac{n_*^2(D_{++} + D_{--})}{p}, \quad (16)$$

$$\gamma = \frac{n_*(D_{++}Y_+ - D_{--}Y_-)}{p}. \quad (17)$$

Unlike the $\mathbf{B} = 0$ case, in this “new” Ohm’s Law \mathbf{J} appears on both sides of the equation. Nevertheless, it is a linear algebraic equation for \mathbf{J} , and it can be solved to give a relation between \mathbf{J} and \mathbf{E}_{ohmic} using a tensorial conductivity. In other words, the conductivity is no longer isotropic when there is a magnetic field. This is called the Hall Effect. The conductivity parallel to the local magnetic field lines is different from the conductivity perpendicular to the local magnetic field lines. When the \mathbf{B} field is “weak,” the cross-product term can be neglected, and we recover the scalar conductivity case.

F When \mathbf{B} is ignored, the old Ohm’s Law says the electric current vector \mathbf{J} is parallel to the ohmic electric field vector, \mathbf{E}_{ohmic} . What happens when \mathbf{B} is not zero? In particular, what happens when \mathbf{B} is very, very large?

=====

Holding collisional frequency constant, the conductivity normal to the magnetic field line goes down with increasing magnetic field, while the conductivity parallel to the magnetic field is unaffected. In other words, For fixed \mathbf{E} , as \mathbf{B} increases, \mathbf{J}_\perp —the component of \mathbf{J} perpendicular to \mathbf{B} —decreases while \mathbf{J}_\parallel is unaffected. Physically, charged particles tend to orbit around a magnetic field line (the frequency is called its “cyclotron frequency”), and they can switch magnetic field lines only by collisions. As the strength of the magnetic field increases, charged particles could orbit many times about a magnetic field line before suffering field-line-switching collisions. Thus its diffusion velocity perpendicular to magnetic field lines is greatly retarded (because it does not leave a field line until it is bumped off by a collision).

If we hold the magnetic field constant, the tensorial conductivity would approach a scalar conductivity as the collisional frequency is increased. Collisions enhances field line switching. In the limit of very high collision frequency (in comparison to cyclotron frequency), charged particles could out complete a full orbit around a magnetic field line before a

collision occurs. Thus, field line switching happens all the time. Under this situation the conductivity of the medium is isotropic. In this limit, the $\gamma \mathbf{J} \times \mathbf{B}$ term is “higher order.” However, we have kept this term here because it provides results which are consistent from more refined theories.