

ME 451C  
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Week #1

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## 1 Introduction

This is an advanced graduate course in fluid mechanics. The assumption is that all enrolled students are already quite knowledgeable in this subject area. One of the objectives of the course is to look into some commonly accepted assumptions and approximations, and to learn about their physical consequences. In addition, this course offers an opportunity for me to talk about some of my favorite fluid mechanics topics—hopefully with special insights.

### 1.1 Reviews

We shall study fluid mechanics using continuum formulation.

### 1.2 Laws of Nature

Continuum fluid mechanics is based on the following three Laws of Nature:

1. The Conservation of Mass (mass can neither be created nor destroyed),
2. The Conservation of Linear Momentum (Newton's Second Law,  $F = ma$ ),

3. The Conservation of Energy (energy can neither be created nor destroyed).

In order to describe a flow field, we need to agree on the definitions of all the symbols we shall use to describe the flow field, such as density  $\rho$ , velocity  $\mathbf{V}$ , pressure  $p$ , stress tensor  $\boldsymbol{\tau}$ , temperature  $T$ , specific internal energy  $e$ , and heat flux  $\mathbf{q}$ .

### 1.3 Supplementary Information

In continuum fluid mechanics,  $\boldsymbol{\tau}$  and  $\mathbf{q}$  are given by the Navier-Stokes and Fourier “constitutive” relations. The “coefficients” in these relations (viscosity and heat conductivity), are some functions of the state of the fluid, such as its temperature  $T$  and density  $\rho$ .

A single fluid in thermodynamic equilibrium, there are two (intelligently chosen) independent state variables. Hence, any state variable can be expressed in terms of these two chosen variables. For example, we can choose density  $\rho$  and temperature  $T$  as the two independent variables of a gas. Then we should be able to get the pressure  $p$  and the specific internal energy  $e$ —using the equation of state and the caloric equation of state of the fluid in question.

For a perfect gas, we have the *usual* equation of state:

$$p = \rho RT \tag{1}$$

where  $R$  is the gas constant (equals to the universal gas constant  $\mathcal{R}$  divided by the molecular weight  $\mathcal{M}$  of the gas in question), and the *caloric* equation of state:

$$e = \int^T C_v(T')dT' + \text{constant} \tag{2}$$

where  $C_v$  is the specific heat at constant volume. Note that it is not allowed to depend on density whenever Eq.(1) is adopted.

It is important to emphasize that the (usual) equation of state and the caloric equation of state are not totally independent of each other. In other words, you can’t arbitrarily pick them without checking whether they are compatible.

## 2 Speed of Sound

If a fluid is compressed (its  $\rho$  increases), its pressure always increases (to resist the compression). The increase of pressure divided by the increase of density is thus a positive number, and has the dimension of the square of a velocity. This number is a property of the fluid. In fact, it can be considered a state variable. When the compression is done *isentropically*, this number is, by definition, the square of the speed of sound:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \quad (3)$$

For a perfect gas, we have:

$$a = \sqrt{\gamma RT} \quad (4)$$

where  $\gamma = C_p/C_v$  and  $C_p$  is the specific heat at constant pressure.

In steady flow problems, characteristics of *supersonic* and *subsonic* flows are distinctly different. In a steady streamtube, the flow “chokes” at Mach one.

### 2.1 Change of speed of sound with respect to density perturbation

When there is a density perturbation while entropy is held constant, there is a pressure perturbation. It should be obvious that, consistent with this pressure perturbation, there is also a perturbation on all other state variables (except entropy—by definition).

We shall find  $(\partial a/\partial \rho)_s$  of particular interest. Actually, the dimensionless parameter of most interest is:

$$\Gamma \equiv 1 + \frac{\rho}{a} \left(\frac{\partial a}{\partial \rho}\right)_s. \quad (5)$$

## 3 Some useful approximations

### 3.1 Isentropic flows

Under some well-defined condition, the isentropic flow approximation is valid.

For a perfect gas, it is readily shown that:

$$\frac{p}{p_o} = \left( \frac{\rho}{\rho_o} \right)^\gamma \quad (6)$$

Let

$$p = p_o + p' = p_o \left( 1 + \frac{p'}{p_o} \right), \quad (7)$$

$$\rho = \rho_o + \rho' = \rho_o \left( 1 + \frac{\rho'}{\rho_o} \right) \quad (8)$$

If the flow is isentropic and the disturbances are small, we would have:

$$\frac{p'}{p_o} = \gamma \frac{\rho'}{\rho_o}. \quad (9)$$

### 3.2 Irrotational flows

Under some well-defined condition, the irrotational flow approximation is valid.

### 3.3 Incompressible flows

A very useful approximation is the “incompressible” approximation—where  $\rho'$  is simply neglected and density is treated as a constant everywhere. Now, in spite of Eq.(9),  $p'$  for such flow is not zero. If you are unclear on this point, ask me a question in class.

Once the incompressible approximation is adopted, the (usual) equation of state of the gas becomes irrelevant. The energy equation is mostly decoupled from the continuity and momentum equation.

### 3.4 Boundary layers

When Reynolds Number is very large, the flow field can be divided into an inviscid region plus thin boundary layers adjacent to solid surfaces (and other interfaces).

## 4 Home Work Problems

1. Look up the properties of air (under standard condition) in your fluid mechanics textbook. Estimate:
  - a The average distance between air molecules,
  - b The average mean-free-path of the molecules, (should you be surprised by the relative magnitude between this number and the previous number?)
  - c The average number of collisions suffered per second by an air molecule.
2. You have a mixture of perfect gases (each with its own molecular weight  $\mathcal{M}_i$ , and the mass fraction of the individual components (to be denoted by  $X_i$ ,  $i = 1, \dots, I$ ). Find  $R$  as a function of the  $\mathcal{M}_i$ 's.
3. Choose  $\rho$  and  $T$  as our two state variables. Work out a general formula for the specific entropy  $s$ . What is  $(\partial s / \partial \rho)_T$ ? What is  $(\partial s / \partial T)_\rho$ ? What is  $(\partial^2 s / \partial \rho \partial T)$ ? Give the explicit answers under the assumption of a perfect gas (with constant  $R$  and  $C_v$ ).
4. Find  $\Gamma$  for a perfect gas.