ME451C
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Week #2

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Abstract

The perfect gas assumption is very good, and it is used pervasively in compressible gasdynamics. Here we want to talk about some interesting non-perfect gas effects. It would be good if you could read this, and bring questions with you to the lectures.

1 Constraints on equations of state

For a fluid medium in therodynamic equilibrium, the *state* of the medium can be specified by *two* state variables. For example, density ρ (mass per unit volume, *e.g.* kilograms per cubic meter) and temperature T (absolute temperature; *e.g.* Kelvin). Once we are given two (good) state variables, all other state variables can be obtained from some equations of state, in principle. For example, $p = P(\rho, T)$ is one of the most popular (I call it "usual") equation of state. Anther example is $e = E(\rho, T)$ (e denotes specific internal energy), and it is usually called the "caloric" equation of state.

It is useful to know that these two equations of state are not totally independent.

The First Law of Thermodynamics says:

$$de = \delta Q - \delta W \tag{1}$$

and claims that e is a state variable.

We now confine ourselves to *reversible* processses on a tiny glob of this material at a certain thermodynamic state ρ, T. A little bit of heat δQ is added to a (closed) system, whose volume changes a little bit as a consequence. According to the Second Law of Thermodynamcis, the specific entropy s (a state variable) of the tiny glob changes by the following amount:[1]

$$ds = (\frac{\delta Q}{T})_{rev}. \tag{2}$$

The work done by the tiny glob on the rest of the universe is:

$$\delta W = pd(\frac{1}{\rho}). \tag{3}$$

Putting things together, we now have:

$$\boxed{Tds = de + pd(\tfrac{1}{\rho}).} \tag{4}$$

Or,

$$\boxed{Tds = dh - \tfrac{dp}{\rho}.} \tag{5}$$

where $h \equiv e + p/\rho$ is called the specific enthalpy. As is well known, eq.(4) and eq.(5) are applicable to irreversible processes.

Let us rewrite eq.(4) as follows:

$$ds = \frac{de}{T} - \frac{p}{\rho^2 T} d\rho. \tag{6}$$

Now, the magnificient Second Law of Thermodynamics says s is a state variable. In other words, it is possible to find an equation of state for s, $s = S(\rho, T)$. The mathematical implication of this simple claim is straightforward: the right hand side of eq.(6) must be a *perfect differential*!

We rewrite eq.(6) as follows:

$$\begin{aligned} ds &= \frac{1}{T}\left(\left(\frac{\partial E}{\partial T}\right)_\rho dT + \left(\frac{\partial E}{\partial \rho}\right)_T d\rho\right) - \frac{p}{\rho^2 T} d\rho, & (7)\\ &= \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_\rho dT + \frac{1}{T}\left(\left(\frac{\partial E}{\partial \rho}\right)_T - \frac{p}{\rho^2}\right) d\rho. & (8) \end{aligned}$$

[1]It is critical that T in this equation is the *absolute* temperature of the tiny glob.

In other words, we must have:

$$\left(\frac{\partial S}{\partial T}\right)_\rho = \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_\rho, \tag{9}$$

$$\left(\frac{\partial S}{\partial \rho}\right)_T = \frac{1}{T}\left(\frac{\partial E}{\partial \rho}\right)_T - \frac{P}{\rho^2 T} \tag{10}$$

Under the assumption that S is a single value smooth function of ρ and T, then its derivative with respect to ρ and T must not depend on the order of operation. In other words, we must require $\partial^2 S/\partial\rho\partial T = \partial^2 S/\partial T\partial\rho$. Hence, $P(\rho, T)$ and $E(\rho, T)$ must satisfy this requirement.

The most common perfect gas assumption $P(\rho, T) = \rho RT$ thus demands that $E(\rho, T)$ be independent of ρ.

2 How much equation of state do we need?

Let the "characteristic" state of the fluid be denoted by subscript o. So in a fluid flow problem, the nominal state is $\rho_o, T_o, p_o, e_o, s_o$, etc. Let the characteristic flow velocity be denoted by u_o,

The ratio p_o/ρ_o has the dimension of the square of velocity. Thus we expect $\rho_o u_o^2/p_o$ to be an interesting dimensionless parameter.

Now, in the flow field, all state variables will deviate from their characteristic values. If we confine ourselves to "small perturbations," meaning that all state variables are "near" the characteristic point, do we really need equations of state that is valid "everywhere?" The answer is self-evident. If we are expected to stick around the neighborhood of the characteristic state, we just need an approximation valid in the neighborhood of interest. In other words, a Taylor series would do.

Now, $P(\rho, T)$ or $E(\rho, T)$ has two independent variables, so we would need, in principle, to do Taylor Series for both variables. But in many interesting problems, the specific entropy is a constant over the flow field. For this class of problems, it is much more intelligent to pick s as one of the two (preferred) state variables. For example, we would be well advised to look for $P(\rho, s)$.

Confining our attention to $s =$ *constant* problems, a Taylor expansion of $P(\rho, s)$ (with respect to ρ disturbance only) yields:

$$P(\rho, s) = p_o + \left(\frac{\partial P}{\partial \rho}\right)_s (\rho - \rho_o) + \frac{1}{2}\left(\frac{\partial^2 P}{\partial \rho^2}\right)_s (\rho - \rho_o)^2 + \dots \tag{11}$$

Now $(\partial P/\partial \rho)_s$ is always positive and is identified with the square of the velocity of sound a_o^2. Hence the above can be rewritten as:

$$P(\rho, s) = p_o + a_o^2(\rho - \rho_o)\left(1 + \frac{\rho_o}{a_o}\left(\frac{\partial a}{\partial \rho}\right)_s \frac{\rho - \rho_o}{\rho_o} + \ldots\right). \qquad (12)$$

You can now see that in a small neigborhood around the characteristic point in state space, two important parameters show up for constant specific entropy flows: a_o (dimensional), and our new friend (dimensionless) Γ, whoich I had introduced in Eq.(5) of my Week1.pdf notes.

3 Bethe-Zeldovich-Thompson gas

For a perfect gas, we have $(\partial \ln a/\partial \ln \rho)_s = (\gamma - 1)/2 > 0$. And $\Gamma = (\gamma + 1)/2 > 0$. A lot of things we take for granted in conventional gas dynamics depends on $\Gamma > 0$. For example, the generally accepted wisdom that an expansion shock is not thermodynamically allowed.

If we look at the state diagram of a gas, plotting lines of constant temperature and specific entropy on a diagram with pressure as the ordinate and specific volume $(1/\rho)$ as the abscissa. For a perfect gas, the constant T lines are hyperbolas, while the constant s lines are slightly steeper on the left. In other words, if you compress a little bit isentropically, both p and T rise a little bit. Fine.

Now what happens near the "triple point" of a real gas? We know the constant T lines are not hyperbolas anymore. But if we compress a little bit isentropically, we still expect both p and T to rise a little bit. Fine.

Now, what about the sign of $\partial^2 P/\partial \rho^2$ near the triple point? You can see that its sign may be different from the sign it takes in the perfect gas region.

Gases for which $\Gamma < 0$ is called a BZT gas. Note that it takes a pretty big negative $\partial^2 P/\partial \rho^2$ to make Γ negative. We shall study the impact of Γ (particularly its sign) in a neat demonstration problem.

4 One dimensional compressible waves and shocks

We shall proceed without using the perfect gas assumption, and limits its attention to waves and weak shocks for which the entropy rise can be neglected.

5 Governing equations

The governing equations for a one-dimensional unsteady problem is:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0, & (13) \\
\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right), & (14) \\
\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} &\approx 0 & (15)
\end{aligned}$$

where s is specific entropy. Under the assumption that the gas in question has the same equation of state everywhere and that $s \approx s_o$ is a good approximation, then pressure p is (approximately) a function of ρ only. We shall let the viscous term tack along, knowing full well that we are interested in problems with very small viscosity (high Reynolds Number problems).

Let a denote the isentropic speed of sound:

$$a \equiv \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{s=s_o}} = a(\rho) \qquad (16)$$

We can then eliminate p in favor of ρ in eq.(14) as follows.:

$$\frac{\partial p}{\partial x} = a^2\frac{\partial \rho}{\partial x} \qquad (17)$$

Multiplying eq.(13) by K—its value yet to be chosen—and adding the result to eq.(14), we obtain:

$$K(\frac{\partial \rho}{\partial t} + (u + \frac{a^2}{K})\frac{\partial \rho}{\partial x}) + \rho(\frac{\partial u}{\partial t} + (u + K)\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) \qquad (18)$$

We are now ready to choose K: two interesting choices are $K = \pm a$. Using either values, we have the following two equations as alternatives to eq.(13) and eq.(14):

$$\frac{\delta_\pm u}{\delta t} \pm \frac{a}{\rho}\frac{\delta_\pm \rho}{\delta t} = \frac{1}{\rho}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right), \qquad (19)$$

where

$$\frac{\delta_\pm}{\delta t} \equiv \frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}. \qquad (20)$$

are directional derivatives in the "characteristic" directions, one moving to the right, one moving to the left, both with relative velocity of a to the local fluid particle.

We can define a new state variable $A(a, s)$ by:

$$A(a, s_o) = A(a) \equiv \int^{\rho} \frac{a}{\rho} d\rho = \int^{a} \left(\frac{a}{\rho} \frac{d\rho}{da} \right) da. \tag{21}$$

and another new state variable $\Gamma(a, s)$ by:

$$\Gamma(a, s_o) = \Gamma(a) \equiv 1 + \frac{\rho}{a} \frac{da}{d\rho} = 1 + \frac{d \ln a}{d \ln \rho}. \tag{22}$$

Hence,

$$A(a) = \int^{a} \frac{da}{\Gamma(a) - 1}. \tag{23}$$

The two governing PDEs can now be rewritten as:

$$\frac{\delta_{\pm}}{\delta t}(u \pm A) = \frac{1}{\rho} \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x} \right). \tag{24}$$

6 Riemann Invariants

Let us define $R_{\pm}$ as follows:

$$R_{+} \equiv u + A(a), \tag{25}$$

$$R_{-} \equiv u - A(a). \tag{26}$$

We can write eq.(19) as follows:

$$\frac{\delta_{+} R_{+}}{\delta t} = \frac{1}{\rho} \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x} \right), \tag{27}$$

$$\frac{\delta_{-} R_{-}}{\delta t} = \frac{1}{\rho} \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x} \right). \tag{28}$$

When the viscous term is neglected (large Reynolds Number limit), the mathematics tells us something very elegant: R_{+} is a constant along a right running wave, and R_{-} is a constant along a left running wave. Let us define a set of "characteristic coordinates" (ξ, η) as follows:

$$\left(\frac{dx}{dt} \right)_{\xi = constant} = u + a, \tag{29}$$

$$\left(\frac{dx}{dt} \right)_{\eta = constant} = u - a. \tag{30}$$

So, we can conclude that R_{+} =constant along a ξ =constant (right running) characteristics, and R_{-} =constant along a η =constant (left running) characteristics.

7 Right running waves

Consider an initial value problem with $s = s_o =$, and a set of right running wave with $u(x, t = 0) = U(x)$ for $|x| < 1$, and $u = 0$ and $a = a_o$ elsewhere.

We first look at the left running waves (which "carries" no disturbance. We have $R_- = u - A(a) = -A(a_o)$. Hence, we have:

$$u = A(a) - A(a_o) = \int_{a_o}^{a} \frac{da'}{\Gamma(a') - 1} \approx \frac{a - a_o}{\Gamma(a_o) - 1}. \tag{31}$$

Solving for a, we have:

$$a = a_o + (\Gamma_o - 1)u. \tag{32}$$

Putting everything together in the R_+ equation, we obtain:

$$\frac{\partial u}{\partial t} + (a_o + \Gamma_o u)\frac{\partial u}{\partial x} = \frac{\nu_o}{2}\frac{\partial^2 u}{\partial x^2}, \qquad (\nu_o = \frac{\mu_o}{\rho_o}). \tag{33}$$

If we move to a coordinate system (X, τ) traveling with the undisturbed speed of sound moving to the right:

$$X = x - a_o t, \tag{34}$$
$$\tau = t, \tag{35}$$

we obtain the famed Burger's equation:

$$\boxed{\frac{\partial u}{\partial \tau} + \Gamma_o u \frac{\partial u}{\partial X} = \frac{\nu_o}{2}\frac{\partial^2 u}{\partial X^2}.} \tag{36}$$

We see that Γ_o is the only interesting parameter that comes through—in addition to ν_o, which just tacked along.

7.1 Perfect Gas

For a perfect gas (with constant specific heats), it is easy to show that:

$$\left(\frac{a}{a_o}\right) = \left(\frac{\rho}{\rho_o}\right)^{\frac{\gamma-1}{2}} \tag{37}$$

where γ is the ratio of specific heats. It is readily shown that for this special case $\Gamma - 1 = (\gamma - 1)/2$. We thus have:

$$A(a) \equiv \int^{\rho} \frac{a}{\rho} d\rho = \int^{a} \frac{da}{\Gamma(a) - 1} = \frac{2a}{\gamma - 1}, \quad \text{(perfect gas)} \tag{38}$$

and Γ is a constant:

$$\Gamma = \frac{\gamma + 1}{2}. \tag{39}$$

Hence, for perfect gases, $\Gamma > 0$. However, there is no thermodynamic constraint on the positiveness of Γ. For non-perfect gases, Γ may be negative under some conditions, and many interesting and novel things can happen then.

8 Interesting properties of the Burger's equation

Without loss of generality, let us assume that at $t = 0$ we have $U(X) > 0$ for $0 < X < 1$ and $U(X) < 0$ for $-1 < X < 0$, and $U = 0$ at $X = -1, 0$, and 1.

It would seem (tentatively) that $u(X, t)$ would remain at zero at $X = -1, 0$, and 1.

8.1 Conservation of area of a "lob"

Let K denote the area under the curve $u(x, t)$ between two zero crossing points of u at any point in time:

$$K \equiv \int u(x, t) dx. \tag{40}$$

It is totally straightforward to prove using eq.(36)—keeping track and taking care of the viscous term on the right hand side properly—that K is independent of time in the small viscosity limit (high Reynolds number)—assuming that no "shock" exists. Thus, for our problem with our initial condition, we have:

$$K_+ = \int_0^1 U(X) dX = \int_0^1 u(X, t) dX, \quad \text{(the positive lob)}, \tag{41}$$

$$K_- = \int_{-1}^0 U(X) dX = \int_)^1 u(X, t) dX, \quad \text{(the negative lob)}. \tag{42}$$

9 Rotation of a straight line

What happens to a straight line initial condition $U(X) = \alpha_o X$?

If we ignore the viscous term, this problem can be solved exactly by separation of variables. We let:

$$u(X,t) = \alpha(t)X, \tag{43}$$

and readily obtain the following exact analytical solution:

$$\alpha(t) = \frac{\alpha_o X}{1 + \alpha_o \Gamma_o t}. \tag{44}$$

It is seen that, as time marches on, the line remains a straight line, but its slope changes with time. And the *sign* of Γ_o is all important in the direction of *rotation*.

10 Distortions of $u(x,t)$ with time

At any moment in time, $u(x,t)$ may be imagined to be made up of many short straightline segments. Thus each short straightline segment is expected to rotate.

Consider what happens when Γ_o is positive. A line segment with positive α_o rotates to reduce its slope. Fine. However, a line segment with negative α_o rotates to *increase* its slope. In fact, at precisely $t = -1/(\alpha_o \Gamma_o) > 0$, the slope goes infinite!

What happens?

Of course, this is the reason we have been patiently keeping the viscous term tacking along. As the slope goes berserk, the viscous term comes in to keep things in check. And a very thin moving normal shock comes in to make everybody happy.

And we will show in class that the normal shock will move. For the $\Gamma_o > 0$ case, the positive lob will extend between $X = 0$ to some $X > 1$ after the shock forms. But nevertheless, the value of K_+ remains constant.

11 Homeworks (due Jan. 18, 2005)

1. Find σ, β of the following tranformation:

$$\xi = \sigma X, \tag{45}$$

$$\eta = \tau, \tag{46}$$

$$W = \beta u, \tag{47}$$

so that the governing equation for W is:

$$\frac{\partial W}{\partial \eta} + W\frac{\partial W}{\partial \xi} = \nu_o \frac{\partial^2 W}{\partial \xi^2}. \tag{48}$$

In other words, Γ_o has been absorbed. This is the standard form of the Burger's equation.

2. Now consider the change of dependent variable from $W(\xi, \eta)$ to $V(\xi, \eta)$:

$$W = \frac{\partial V}{\partial \xi}. \tag{49}$$

Derive the PDE for V, which allows one simple integration. Now perform the further change of dependent variable:

$$\Omega = \exp(-\frac{V}{c}) \tag{50}$$

where c is a constant. This is the famed Cole-Hopf tranformation. Show that the governing equation for Ω is the linear heat conduction equation by choosing a special value for c.

3. Assume $\Gamma_o > 0$. Show that for the right running wave problem, the positive and negative lobs eventually form a N-wave, and the width of the N-wave grows with time for large t as the square root of t.

4. How does the strength of the shocks (the jumps in u) of the N-wave evolve with time for large time?

5. What happens—for the right running wave problem with the same initial condition—when Γ_o is negative? In particular, will K_+, K_- be constants with respect to time? Qualitative answer will do.

6. Use a coordinate system (ξ, η) moving with a constant speed U_{sh}:

$$\xi = X - U_{sh}t, \tag{51}$$

$$\eta = \tau, \tag{52}$$

and consider *steady* (independent of η) problems in this coordinate system. You can solve the resulting ODE analytically. What kind of problems do the solutions describe? (assume $\Gamma_o > 0$; talk a little about $\Gamma < 0$).

I expect to cover some of these problems in class. But I think they are good exercises. Please let me know whether my assignments are too easy or too much work.