

ME 451C
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Week # 9

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Abstract

We have learned that certain body forces (electrostatic and electromagnetic) can influence the diffusion processes, and so can fast chemical reactions. We shall show here that the convection term—inside the substantial derivative—can also have a strong influence on the diffusion processes. Of course, everybody knows that “turbulence” enhances diffusion (of momentum and everything else). Nevertheless, it is interesting to see a very simple example where the convective term strongly enhances the (laminar) diffusion process.

And we will talk about the RNG theory of turbulence.

1 Convection-Diffusion Coupling

Consider the following generic convective-reactive-diffusive PDE:

$$\frac{\partial \mathbf{X}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{X} = \mathbf{W} + \nabla \cdot D \nabla \mathbf{X} \quad (1)$$

where \mathbf{X} and $\mathbf{W}(\mathbf{X})$ N -dimensional column vectors, \mathbf{v} is the mixture fluid velocity and ∇ and $\nabla \cdot$ are gradient and divergence operators in physical space. Here, \mathbf{W} represents chemistry. For the sake of simplicity, we have assumed that the diffusion matrix is diagonal and all diffusion coefficients are the same. We have written out the substantial derivative on the left hand side in full—an unsteady term and a convective term.

1.1 Time scales

The physical dimension of Eq.(1) is, loosely speaking, \mathbf{X}/time . So, the most sensible chemical time scales for the chemistry term are the reciprocal of the eigenvalues of the Jacobian of \mathbf{W} with respect to \mathbf{X} . There are N eigenvalues, so there are N time scales (A of them are infinity for chemistry problems). We shall denote them by τ_n , ordered in ascending magnitude of its real part.

What about time scales for convection and for diffusion? Both terms involves the gradient operator and also the divergence operator.

For the convection term, its time scale is usually called the *residence time*." Once the physical problem is posed, this is usually obvious.

For the diffusion term, we need to know a characteristic diffusion length, L_d . The diffusion time scale is then estimated as L_d^2/D . The tricky issue is how to identify the characteristic diffusion length. Given a PDE where the diffusion term is "needed" to playing a significant role, the relevant characteristic diffusion length is *determined* by the requirement that the diffusion term is "balanced" by some other term in the equation. If the some other term is the convection term, then the characteristic diffusion length is the thickness of the boundary layer (for high Reynolds Number flows). If the some other term is a chemistry term, then the characteristic diffusion length is such that the diffusion time scale "balances" the relevant chemistry time scale. As we recall from our experience in solving linear diffusion problems by separation of variables, the diffusion PDE has infinite number of time scales.

The ratio of either convection or diffusion time scales to chemical time scale is called a *Damkohler Number*. So, in a convection-reaction-diffusion problem, there are N Damkohler Numbers. A major consequence is that Mach Number and Reynolds Number are no longer adequate to fully characterize a flow. We now have N Damkohler Numbers to deal with.

We shall look at an interesting problem with a special velocity flow field \mathbf{v} . We will let the chemistry term tag along, but eventually we shall drop the chemistry term to allow more detailed analysis of the strong coupling between convection and diffusion with this velocity flow field.

1.2 Unsteady 2-Dimensional incompressible stagnation point

We all know that turbulence enhances diffusion. And this observation is totally reasonable intuitively.

Let us consider a toy problem with molecular diffusion only, but in an unsteady flow. We consider a 2-Dimensional unsteady stagnation flow in (x, y) with velocity $\mathbf{v} = (u, v)$:

$$u = x\Omega \cos(\omega t), \quad (2)$$

$$v = -y\Omega \cos(\omega t). \quad (3)$$

where Ω and ω are constants and have the physical dimension of reciprocal time. It is easy to verify that they satisfy the incompressible continuity equation. It is the unsteady flow about a stagnation point located at the origin of the x, y plane.

Our PDE is now (assuming D to be a constant):

$$\frac{\partial \mathbf{X}}{\partial t} + \Omega \cos(\omega t) \left(x \frac{\partial \mathbf{X}}{\partial x} - y \frac{\partial \mathbf{X}}{\partial y} \right) = \mathbf{W} + D \left(\frac{\partial^2 \mathbf{X}}{\partial x^2} + \frac{\partial^2 \mathbf{X}}{\partial y^2} \right). \quad (4)$$

We shall show with this toy problem that a convection term can introduce very interesting complications.

1.3 Coordinate transformation

Eq.(1) is written using Euler formulation. The following is a standard transformation to recast it in Lagrangian formulation:

$$\tau = t, \quad (5)$$

$$\xi = x \exp \left(-\frac{\Omega}{\omega} \sin(\omega\tau) \right), \quad (6)$$

$$\eta = y \exp \left(+\frac{\Omega}{\omega} \sin(\omega\tau) \right) \quad (7)$$

The coordinates (ξ, η) are Lagrangian in the sense that it is identified with a fluid “particle.” In fact, they denote the coordinate of the fluid particle of interest at time $\tau = 0$.

Eq.(4) becomes:

$$\frac{\partial \mathbf{X}}{\partial \tau} = \mathbf{W} + D \left(\frac{\partial^2 \mathbf{X}}{\partial \xi^2} \exp \left(-\frac{2\Omega}{\omega} \sin(\omega\tau) \right) + \frac{\partial^2 \mathbf{X}}{\partial \eta^2} \exp \left(+\frac{2\Omega}{\omega} \sin(\omega\tau) \right) \right) \quad (8)$$

We see that the “effective” diffusion coefficients in ξ, η space are time dependent. The diffusion coefficient in the ξ -direction is 180° out of phase with the diffusion coefficient in the η direction. Both effective coefficients can be much larger than D when Ω/ω is a moderately large number. Of course, the diffusion coefficients in x, y space are totally ordinary and nothing special.

Note that the chemistry term \mathbf{W} is untouched by the turmoil of the coordinate transformation. But remember, if you find \mathbf{W} to be stiff and want to do reduced chemistry on it, your diffusion terms will need to be treated with care.

1.4 What happens physically

What happens is that this stagnation point flow strongly influences the relevant characteristic length for diffusion. A lump of nonuniformity (such as a circle of radius L) becomes a thin sliver of nonuniformity (by the action of convection) when Ω/ω is moderately large, and the thin sliver is readily be smeared out by (molecular) diffusion. On the other half of the cycle when the thin sliver is coming back to be a circle, the diffusion process is then very slow because of the much bigger characteristic diffusion length.

2 RNG theory of turbulence

RNG stands for *Renormalization Group* theory, sometimes it is called RG instead of RNG. It originated in quantum field theory, a procedure “to remove divergences within the framework of perturbation theory.” It was successfully used by Ken Wilson to describe critical phenomena in phase transitions of the second kind [1], and won a Nobel Prize.

The methodology was first applied to turbulence in the 1980s by Edward Teodorovich of Russia. Later Victor Yakhot and Steven Orszag [3] of U.S.A made significant progress, and obtained very good results which stimulated strong interests in the turbulence community. I personally knew many active scholars in this arena, both in Russia and in the U.S., and have had the opportunity to discuss RNG with them in great details. After much study, I remain a skeptic on RNG as a methodology for turbulence, but I do not dispute that it has generated very good results.

I wrote a paper putting forth an explanations of why good results

were obtained, and pointed out some of the unconventional steps taken by RNG advocates in obtaining these results. The paper is posted on the course webpage as supplementary reading for this week. I will talk about it in class.

Homework

1. For our toy problem of unsteady two-dimensional stagnation point flow, let's consider the no chemistry case $\mathbf{W} = 0$ with a scalar unknown $\mathbf{X} = X$. In addition, we limit ourselves to the one-dimensional unsteady case so that $X = X(\xi, \tau)$ only. Find a transformation to transform this PDE into the usual constant coefficient unsteady one-dimensional diffusion PDE.
2. Consider the toy problem in §1 of Week #8 Notes: Eqs.(1,2,3) and Eqs.(12,13) with $w_2 = w_3 = 0$ and $D_{11} \neq D_{22} \neq 0$. We shall consider an unsteady one-dimensional (y, t) problem—the domain of interest is $0 \leq y \leq L, t \geq 0$. The initial conditions are:

$$X_1(y, 0) = c_1(y), \quad (9)$$

$$X_2(y, 0) = c_2(y). \quad (10)$$

The boundary conditions at $y = 0$ and $y = L$ are:

$$X_1(0, t) = b_1(t), \quad (11)$$

$$X_2(0, t) = b_2(t), \quad (12)$$

$$D_{11} \frac{\partial X_1}{\partial y}(L, t) = 0, \quad (13)$$

$$D_{22} \frac{\partial X_2}{\partial y}(L, t) = 0. \quad (14)$$

Assume that $c_1(0) \neq b_1(0), c_2(0) \neq b_2(0)$.

- a Write down the leading order reduced chemistry PDE for X_1 and X_2 in the limit of asymptotically small ϵ . Where in our domain of interest does this PDE apply? By dimensional analysis, there are two characteristic diffusion time scales: L^2/D_{11} and L^2/D_{12} . What can you say about the general qualitative picture of the solution for t small in comparison to these values, and for t of this order of magnitude or larger?

- b** Find the initial conditions for this reduced chemistry PDE.
- c** Find the boundary condition at $x \approx 0$ for this reduced chemistry PDE. Find the ODE governing the structure of the thin layer in which X_1 is not approximately equal to X_2^2 (remember, in the slow subspace, the decay of the fast modes provide M algebraic equation; similarly, in the fast subspace, the slow modes provides $N - M$ approximate conserved scalars). How does the thickness of this layer scale with ϵ ?
- d** Find the boundary condition at $y \approx L$ for this reduced chemistry PDE.
- e** This is *not* required. But if you have in your toolbox a diffusion PDE solver that can handle the reduced chemistry PDE, I would appreciate getting some numerical solutions from some volunteers. Pick for yourselves $c_1(t), c_2(t), b_1(y)$ and $b_2(y)$ (constants are fine), the values of L, D_{11} and D_{22} , and the range of t the numerical simulations would cover. Remember, the reduced chemistry model is valid for any “asymptotically small” values of ϵ .

References

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- [2] Teodorovich, E., The use of the Renormalization group method to describe turbulence, *Iz. Ross Akad. Nauk. FAO*, **29**, 2, pp.149-163. 1983.
- [3] Yakhot, V. and Orszag, S., Renormalization-group analysis of turbulence, I. Basic theory. *J. Scient. Comput.*, **1**, pp.3-51, 1986.
- [4] Lam, S. H., On the RNG theory of turbulence, *Phys. Fluids A*, **4**, 5, pp.1007-1017, 1992.