The Prandtl Boundary Layer Theory

S. H. (Harvey) Lam

January 25, 2004

Abstract

For Problems that have very large characteristic Reynolds Numbers (R_e), the legendary Prandtl boundary layer theory is a must.

1 The Major Claims

It is understood that Reynolds Number represents the ratio of "characteristic" inviscid stresses (ρU^2) and "characteristic" viscous stresses ($\mu U/L$).

The following are the major claims:

1. In the large R_e limit, all viscous terms (including heat conduction) can be neglected (for certain problems). This yields the theory of inviscid flows—and is valid **away** from boundaries and interfaces. Provided there is no flow separations, the static pressure acting on solid boundaries can be computed with excellent accuracy from the inviscid theory.

2. In the immediate vicinity of solid boundaries, a thin layer, called the boundary layer exists—in which the viscous terms must be respected. The characteristic thickness δ of this boundary layer is inversely proportional to some fractional power of R_e—it approaches zero thickness in the limit of asymptotically large R_e. For laminar boundary layers, $\delta/L \propto \mathrm{O}({R_e}^{-1/2})$.

3. Even though in general the boundary layer adjacent to a curved surface (with finite local radii of curvature) is necessarily curved, the "world" in that thin layer looks flat for anyone sitting inside and looking around. (for the same reason why some people still

think the earth is flat). This observation bring great simplifications! So long as the boundary layer is thin (i.e. $\delta/L << 1$), the Cartesian coordinate system is a very good approximation for writing down the boundary layer equations!

4. So long as the boundary layer is thin, the pressure distribution inside the boundary layer can be well approximated by the inviscid pressure distribution generated by the inviscid theories at the outer "edge" of the boundary layer. (Lets say it again: the pressure inside the very thin layer can **not** be significantly different from the inviscid pressure calculated at the outer edge of the boundary layer—so long as the flow remains attached.)
5. As for the viscous terms, some of them must be kept, and some of them can be neglected, in the boundary layer equations. For example, the second derivative with respect to the streamwise coordinate is expected to be negligible to the second serivative with respect to the normal coordinate.

All of the above claims are intuitively reasonable. They need to be verified by mathematics.

2 The Boundary Layer Equations

We shall limit our attention to two-dimensional problems.

Let u, v denote the velocity components of $\mathbf{V}$ in the x, y directions, respectively. The coordinate system x, y is actually a curvilinear coordinate system, but appears (approximately) as a Cartesian coordinate system within the boundary layer approximation ($\epsilon \equiv \delta/L << 1$).

In terms of the boundary layer coordinates, the conservations laws can be written as follows:[1]

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = O(\epsilon) \qquad (1)$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \underbrace{\frac{\partial \varepsilon_{yx}}{\partial y}} + O(\epsilon) \qquad (2)$$

$$O(\epsilon) = -\frac{\partial p}{\partial y} \qquad (3)$$

[1] The summation convention is not used here. The substantial derivative operator D/Dt is $\partial/\partial t + u\partial/\partial x + v\partial/\partial y$, as always.

$$\rho \frac{D}{Dt}\left(e+\frac{u^2}{2}\right) = -\left(\frac{\partial(up)}{\partial x}+\frac{\partial(vp)}{\partial y}\right) \underbrace{+\frac{\partial(u\varepsilon_{yx})}{\partial y}-\frac{\partial q_y}{\partial y}} +O(\epsilon) \tag{4}$$

where ε_{yx} is the viscous force (per unit area) in the +x direction on a surface element whose normal is in the +y direction, e is internal (thermal and chemical and nuclear ... per unit mass), and q_y is the heat conduction rate (energy per unit area per unit time) in the +y direction. Note that $\varepsilon_{xy} = \varepsilon_{yx}$ (as always), but ε_{xy} played no role in the boundary layer equations.

The equations above have been written in the form that makes it very easy to understand the physical origin of each term. No body force had been included for the sake of simplicity. Note that all the new terms added by viscosity (and heat conduction) have been marked by underbraces.[2]

Multiplying eq.(5) by u, we obtain an equation whose physical dimension is the same as the energy equation (energy per unit mass per unit time):

$$\rho\frac{D(u^2/2)}{Dt} = -u\frac{\partial p}{\partial x} + \underbrace{u\frac{\partial \varepsilon_{yx}}{\partial y}} +O(\epsilon) \tag{5}$$

Subtracting this from eq.(4) (we did this trick before!), we obtain, after a little bit of house-cleaning:

$$\rho\left(\frac{De}{Dt}+p\frac{D}{Dt}(\frac{1}{\rho})\right) = \underbrace{\varepsilon_{yx}\frac{\partial u}{\partial y}-\frac{\partial q_y}{\partial y}} +O(\epsilon) \tag{6}$$

Using the "differential equation of state" that we talked about, we now have the equation for the specific entropy s (entropy per unit mass):

$$\rho T\frac{Ds}{Dt} = \underbrace{\varepsilon_{yx}\frac{\partial u}{\partial y}-\frac{\partial q_y}{\partial y}} +O(\epsilon) \tag{7}$$

The first term on the right hand side is the so-called *viscous dissipation function* under the boundary layer approximation.

[2]It is understood that these equations need to be supplemented by an equation of state and formulas for the materical properties.

The boundary layer equations as written above are valid for both laminar and turbulent flows. For laminar flows, the Navier-Stokes viscous stress tensor and the Fourier Law of heat conduction are adequately approximated by:

$$\varepsilon_{yx} = \mu \frac{\partial u}{\partial y} + O(\epsilon), \quad (8)$$

$$q_y = -\kappa \frac{\partial T}{\partial y} + O(\epsilon). \quad (9)$$

The second coefficient of viscosity is a no show in Prandtl's boundary layer equations.

We assume all solid walls are non-porous. Then the noslip condition is applied at all solid walls. A boundary layer solution is completely specified if the velocity at the edge of the boundary layer $U_e(x)$ is given (along with appropriate initial conditions). It is intuitively reasonable that you need to specify some "upstream" condition as the "initial" condition of the boundary layer.

For turbulent flows, we will need to come up with some alternative descriptions—usually empirical—that make sense from the dimensional analysis point of view. Remember, we know the dimensions of ε and q_y, and those of μ and κ also.

2.1 Laminar Case and R_e Effects

The x-momentum equation and the energy equation for laminar boundary layers are written explicitly below:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \underbrace{\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)} + O(\epsilon) \quad (10)$$

$$\rho \frac{D}{Dt}\left(e + \frac{u^2}{2}\right) = -\left(\frac{\partial (up)}{\partial x} + \frac{\partial (vp)}{\partial y}\right) \quad (11)$$

$$\underbrace{+\frac{\partial}{\partial y}\left(u(\mu \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right)} + O(\epsilon) \quad (12)$$

OK. Does the solutions of the laminar boundary layer equations depend on the characteristic Reynolds Number R_e of the problem?

One could say yes, or one could say no. It depends on what you mean by a solution. Let's do the homework problem (#2) and see.

3 Similar Solutions for Laminar Boundary Layers

Suppose we have:

$$U_e(x) = ax^m, \tag{13}$$

where a and m are constants. How much can we deduce with dimensional analysis?

It is obvious that m is dimensionless (dimensional exponents make no sense); only a is dimensional—its dimension is velocity divided by length to the m-th power. Most importantly, the problem as specified has no characteristic length. Except for the case of $m = 0$ (when a is velocity) we do not have a characteristic velocity. In any case, we have only one dimensional parameter a when we normally expect two dimensional parameters. Therefore, we have one less dimensionless parameters.

A big deal in fluid mechanics is that we usually deal with PDEs (partial differential equations). If somehow a problem can be shown to be governed by ODEs (ordinary differential equations), it is good news indeed.

Normally we nondimensionalize x by L and y by δ. If a problem provided no L, the theoretical relations between dimensionless parameters must not involve x/L. Aha! Two independent variables have been reduced to one! ODEs, here we come! The solution (independent of x/L) obtained is called a similarity solution.

So whenever we have a problem without characteristic length, the main task at hand is to find δ (a length, which may be x-dependent) in terms of all the other dimensional parameters.

For the class of problems associated with eq.(13), check out the physical dimension of $\sqrt{\frac{\mu}{\rho a}} x^{\frac{1-m}{2}}$. It is indeed the recommended choice for $\delta(x)$.

The case $m = 0$ is the boundary layer on a sharp flat plate at zero angle of attack. The solution is called the Blasius solution. The case $m = 1$ is the boundary layer at the stagnation point of any (two-dimensional) blunt body. The solution is affectionately called the stagnation point solution. The rest of the solutions for other values of m are called the Falkner-Skan solutions.

4 Momentum Integral

Read the derivation of the Momentum Integral equation in §4-6.2 (you may also want to read §4-6.3), and make yourself feel comfortable with the derivation. The derived equation (Eq.(4-122)) is exact (nothing was discarded or dropped from the original boundary layer equations), but notice that it is one scalar equation for three unknowns, C_f, θ and H. So one is obligated to do something in order to get a theory out of this one equation. Take a look at §4-6.6, the Thwaites Method. Essentially, Thwaites obtained some correlations of the "unknowns" using known solutions and used them to help out. His answer (Eq.(4-137)) is very simple to use.

An interesting observation is: does the momentum integral equation hold when the boundary layer is turbulent? One can verify, by checking through the derivation, that the answer is yes. Analogous to the laminar case, one needs to somehow provide the missing equations (or correlelations).

5 Unsteady Flows

You have a plane horizontal solid surface in contact with a viscous fluid at rest. At t=0, the solid surface suddenly start moving in the x-direction (along the plane surface). By viscous shear stress and the no slip condition, the fluid (in the $y > 0$ space), will be disturbed. What is the thickness of the disturbed region? The is called the Stokes, problem. You can take a good crack at the question using dimensional analysis.

Close your mouth, and pay attention to how you normally breath. You slowly inhale, and slowly exhale. Think of a small parcel of fresh air just outside your nose before you inhale. Now inhale. How far down do you think this parcel of fresh air go before you start exhaling? In other words, how does your lung ever get to meet the fresh air outside your nose? Think about this cute (viscous flow) problem. Your life actually depends on it. (it is not a high Reynolds Number problem).

6 Three Dimensional Effects

You put some tea leaves in your cup of water. You stir the tea with a spoon. You had put too much tea, and some had settle down on the bottom of the cup. You notice that the tea leaves are collected near the center of the cup. Why? This is also a cute problem to think about.

7 Homework

Some of the problems I may work on during my lecture. But I want all to be able to work through them independently.

Homework 1: Look up the constant properties (ρ, μ and κ) Navier Stokes equation for cylindrical coordinates from White. Note that there are terms which came from the non-Cartesian (but curvilinear) coordinates. Consider the steady, two-dimensional flow over a circular cylinder with dimensional radius r_* by an otherwise uniform flow of velocity U_*. Introduce the following dimensionless variables:

$$X = \theta, \quad Y = \frac{r - r_*}{\delta} \tag{14}$$

$$U = \frac{u}{U_*}, \quad V = \frac{v}{U_*}\frac{r_*}{\delta}, \quad P = \frac{p - p_*}{\rho U_*^2} \tag{15}$$

$$\frac{\delta}{r_*} = \text{some function of } R_e, \quad R_e = \frac{\rho U_* r_*}{\mu}. \tag{16}$$

The hypothesis is the dimensionless variables (X, Y, U, V, P are all $O(1)$ entities, while $\delta/r_* << 1$ when R_e is some large number. (a) Can you convince yourself that all the non-Cartesian terms in the continuity equation and the streamwise momentum equation becomes negligible in the $\delta/r_* \to 0$ limit. In other words, the original equations in cylindrical coordinates now look like Cartesian coordinates (in this limit). (b) If you believe that the viscous terms must be competitive in this thin layer, what would you pick for $\delta(R_e)$? (c) What would you call δ in English?

Homework 2: There is some mechanical engineering designer interested in buying (with good money) steady laminar boundary layer solutions for a certain flow problem at 150 different Reynolds Numbers (all large). The geometry of the problem is fixed, but the size (i.e. scale), the fluid (air, water, or oil), the flow velocity, are yet to be decided. How many solutions do you need to obtain in order to open a profitable consulting relationship with him/her? Explain.

Homework 3: Look up your inviscid theory fluid books and notes to find the inviscid (irrotational) solution for the steady flow over a cylinder (with or without circulation). Find $\partial p/r\partial\theta$ and $\partial p/\partial r$ on the surface of the cylinder. Note that they are "of the same order of magnitude." How do you reconcile this observation with the conclusion that $\partial P/\partial Y$ is neglible in comparison to $\partial P/\partial X$ in the large R_e limit?

Homework 4: You are interested in the boundary layer near the stagnation point of a uniform flow over a circular cylinder (without circulation). Obviously, the $m = 1$ stagnation point solution is right there for you to use! However, you need the value of a. Find the theoretical value for a using the inviscid theory you have found for the previous problem. (Hint: you are a tiny observer, sitting in the thin boundary layer near the front stagnation point of the cylinder, and look upstream ahead. What inviscid flow do you see coming toward you?)