Rayleigh's Inflexion Point Theorem

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Abstract

White did not present in his book the details of Rayleigh's Inflexion Point Theorem for the inviscid Orr-Sommerfeld Equation. I have not looked up the original Rayleigh papers or the important followup papers by others as quoted by White. The following notes come from my own messing around, and is likely an approximation to what were done in those papers. I assume you would find it interesting.

1 The Inviscid Orr-Sommerfeld

The Inviscid Orr-Sommerfeld Equation is:

$$V'' - \alpha^2 V - \frac{U''}{U - c} V = 0 \tag{1}$$

where $U(\eta)$ is the velocity profile under scrutiny ($\eta = y/\delta$; with $U(\eta \geq 1) = 1$ assumed), α is a (given) real number, and c is the eigenvalue to be found.[1]

For $\eta > 1$, eq.(1) can readily be solved analytically to yield $V \propto \exp -\alpha\eta$. Thus we have:

$$V'(1) = -\alpha V(1). \tag{2}$$

Since we have a linear problem, we can arbitrarily set $V(1) = 1$ (or any constant). Being inviscid, the boundary condition at the solid

[1] δ is dimensional boundary layer thickness, and is Reynolds Number dependent. For example, $\delta/x \approx 5/\sqrt{Re_x}$ for the Blasius profile.

wall ($\eta = 0$) is $V(0) = 0$ (no flow through the solid wall). We have no right to impose the no-slip condition.

It is obvious that $V(\eta) = 0$ is an exact (trivial) solution. The job on hand is to find $c(\alpha)$ (complex!) such that a non-trivial solution exists.

2 Preliminary manipulations

Let's denote complex conjugate by an overbar. So, the complex conjugate of V—i.e. $(V_r + iV_i)$—is $\bar{V}$—i.e. $(V_r - iV_i)$. We denote the magnitude of a complex number as follows:[2]

$$|V| = \text{magnitude of } V \equiv \bar{V}V = V_r^2 + V_i^2. \tag{3}$$

Now we multiply eq.(1) by $\bar{V}$ to obtain:

$$\bar{V}V'' - \alpha^2|V| - \frac{U''}{U-c}|V| = 0 \tag{4}$$

Next we integrate over the whole boundary layer, from $\eta = 0$ to $\eta \to \infty$. The first term is "massaged" by integration by parts to yield:

$$[\bar{V}V']_{\eta=0}^{\eta\to\infty} = \int_0^\infty \left[|V'| + \left(\alpha^2 + \frac{U''}{U-c}\right)|V|\right] d\eta = 0 \tag{5}$$

Imposing the (homogeneous) boundary conditions, the left hand side vanishes (note: the upper limit is $\eta \to \infty$ and not $\eta = 1$). We then have:

$$\int_0^\infty \left[|V'| + \left(\alpha^2 + \frac{U''}{U-c}\right)|V|\right] d\eta = 0 \tag{6}$$

It is useful to remind ourselves that c is a complex number. Note that except for the term involving a fraction, the two other terms in the integrand are always positive.

Multiplying the numerator and the denominator of the fraction term inside the bracket by $U - \bar{c}$, we have:

$$\int_0^\infty \left[|V'| + \left(\alpha^2 + \frac{(U-\bar{c})U''}{|U-c|}\right)|V|\right] d\eta = 0. \tag{7}$$

[2]This is probably not the common notation—which would say $|V| \equiv \sqrt{V_r^2 + V_i^2}$. So be warned. This is my notation.

The real and imaginary component of this complex equation are:

$$\int_0^\infty \left[|V'| + \left(\alpha^2 + \frac{(U-c_r)U''}{|U-c|}\right)|V|\right] d\eta = 0, \tag{8}$$

$$c_i \int_0^\infty \frac{U''}{|U-c|}|V| d\eta = 0. \tag{9}$$

Remember, we are looking for c_r and c_i such that a non-trivial $V(\eta)$ exists.

3 Rayleigh's Theorem

Rayleigh's Inflexion Point Theorem follows from eq.(9):

> If $U''(\eta)$ does not change sign inside the boundary layer (i.e. have an inflexion point), c_i must be zero.

This is a sufficient condition to conclude that c_i must be zero (i.e. the profile under study is stable), but not a *necessary condition*. On the other hand, $U''(\eta)$ changes sign inside the boundary layer is a necessary condition for $c_i \neq 0$ (the boundary layer can be either stable or unstable), but it is not a sufficient condition.

The above was presented in class. The following was not.

Now, if $c_i \neq 0$, then the integral in eq.(9) must be zero. For this case, the value of c_r in the numerator of the fraction term in eq.(8) can be arbitrarily changed (this is clever!). Let's change that c_r into U_{PI}, the "PI" subscript indicates it is U evaluated at the "point of inflexion." It is now clear that (when $c_i \neq 0$ is assumed) then

$$(U - U_{PI})U'' < 0 \tag{10}$$

must be true somewhere inside the boundary layer (note: both factors changes sign at the inflexion point; so this inequality wants the product to keep a negative sign on *both* sides of the inflexion point) so that the messy fraction term in eq.(8) is negative. This is another necessary condition for $c_i \neq 0$—just having an inflexion point inside the boundary layer is not enough—or *sufficient*.

4 Closing Remarks

When analytical solutions are not available, finding eigenvalues is a guessing game. Theorems such as Rayleigh's Theorem are helpful in the guessing game. Rayleigh did not have Matlab.