A Robust Controller for Scalar Autonomous Optimal Control Problems

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Abstract

Is it possible to exert control on a physical system without *detailed* knowledge of its open-loop dynamics? Recently, Lam [1, 2] showed that a special dynamic control law can accomplish this feat for a certain class of nonlinear systems, provided accurate and good quality data of all output variables, including their time derivatives, are assumed available to the controller. This paper generalizes this theoretical idea to a class of *scalar autonomous* optimal control problems.

1 Introduction

The recent explosive advances of microelectronics provide powerful new tools to control engineers. One can readily assume that ample computational power is available to the (programmable) controller, and that the sampling time, t_s, is "sufficiently small." In addition, advances in sensor technologies make it possible to also assume that very accurate direct measurements of the output variables are available to the controller, *including* their time derivatives. On the other hand, the task of formulating and validating a mathematical model for a real world system has remained as difficult as ever. An interesting question is: can we exploit the advances of microelectronics and sensor technologies to alleviate the modeling difficulties? This question was addressed by the author [1, 2] for classical control problems, and the answer reached was affirmative—provided "approximate" solutions are considered acceptable. With the new approach, *detailed* knowledge of the system's open-loop dynamics is not needed so long as accurate direct measurements of the output variables *and* their time derivatives are available to the controller. This paper extends this theoretical idea to deal with a general class of *scalar autonomous* optimal control problems.

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2 Statement of the Problem

Consider the optimal control problem for the scalar variable x governed by the following quasi-linear, autonomous ODE:

$$\frac{dx}{dt} = g(x) + bu \tag{1}$$

where $g(x)$, a bounded differentiable function, represents the open-loop dynamics of the physical system, b is a constant, and u is the scalar control variable. The initial and terminal conditions are:

$$\begin{aligned} x(0) &= x_o, && \text{(2a)} \\ \frac{dx}{dt}(t_f) &= 0, \quad t_f >> 1. && \text{(2b)} \end{aligned}$$

Of particular interest is the special case when the time interval of interest is semi-infinite, *i.e.* when t_f is asymptotically large [3]. The mathematical problem is to find $u(x;t)$ such that the resulting trajectory $x(t)$ generated by (1) *as an initial-value problem* not only honors both (2a) and (2b), but also minimizes the *total cost* $\mathcal{J}$ of the whole endeavor defined by:

$$\mathcal{J}(x_o, x_f; t_f) \equiv \int_0^{t_f} \mathcal{L}(u(t), x(t))dt, \tag{3}$$

where

$$\mathcal{L}(u, x) = \phi(x) + \frac{u^2}{2}, \tag{4}$$

$\phi(x)$ is any user-defined, non-negative, bounded and differentiable function of x, and the term $u^2/2$ is included to reflect the usual desire to minimize control efforts. It is important to note that both $g(x)$ and $\mathcal{L}(u, x)$ have been assumed autonomous, *i.e.* they are not allowed to have explicit time dependence. It is assumed that x, u, g and ϕ have been appropriately normalized such that they are nominally $O(1)$. The independent variable t is also assumed normalized so that the characteristic time of the problem—appropriately defined—is $O(1)$. The sampling time t_s of the controller is assumed to be "asymptotically" small in comparison.

The terminal condition (2b) implies that x monotonically approaches a constant value, to be denoted by $\widehat{x}_0$, for $t >> 1$. The associated value of u, to be denoted by $\widehat{u}(\widehat{x}_0)$ or simple $\widehat{u}_0$, can be solved for by substituting (2b) into (1). We obtain:

$$\widehat{u}_0 \equiv \widehat{u}(\widehat{x}_0) = -\frac{g(\widehat{x}_0)}{b}. \tag{5}$$

The value of $\mathcal{L}(\widehat{u}(\widehat{x}_0), \widehat{x}_0)$, to be denoted by $\widehat{\mathcal{L}}(\widehat{x}_0)$ or simply $\widehat{\mathcal{L}}_0$, is then:

$$\widehat{\mathcal{L}}_0 \equiv \widehat{\mathcal{L}}(\widehat{x}_0) = \phi(\widehat{x}_0) + \frac{g^2(\widehat{x}_0)}{2b^2}. \tag{6}$$

For large but finite t_f, (3) can be rewritten as:

$$\mathcal{J}(x_o; t_f; \widehat{\mathcal{L}}_0) = t_f \widehat{\mathcal{L}}_0 + \Delta\mathcal{J}(x_o; \widehat{\mathcal{L}}_0) \tag{7}$$

where $t_f \widehat{\mathcal{L}}_0$ represents the cost of simply staying put on $\widehat{x}_0$ for all $0 \le t \le t_f$, and $\Delta\mathcal{J}(x_o; \widehat{\mathcal{L}}_0)$ represents the finite cost of transition from $x = x_o$ to $x = \widehat{x}_0$, and is given by

$$\Delta\mathcal{J}(x_o; \widehat{\mathcal{L}}_0) = \int_0^\infty \left[\mathcal{L}(u(t), x(t)) - \widehat{\mathcal{L}}_0\right] dt. \tag{8}$$

For problems with $t_f >> 1$ and $\widehat{\mathcal{L}}_0 > 0$, the total cost $\mathcal{J}$ is dominated by the $t_f \widehat{\mathcal{L}}_0$ term. Hence, $\widehat{\mathcal{L}}_0$ must be a minimum of $\widehat{\mathcal{L}}(x)$ in order to achieve the lowest cost:

$$\left(\frac{d\widehat{\mathcal{L}}}{dx}\right)_{x=\widehat{x}_0} = 0, \quad \left(\frac{d^2\widehat{\mathcal{L}}}{dx^2}\right)_{x=\widehat{x}_0} > 0, \tag{9}$$

The first equation in (9) provides an algebraic equation for $\widehat{x}_0$. For the moment, we assume at least one root exists. If more than one root exist, all roots are potentially of interest if t_f is finite since the choice of root can depend on the values of x_o and t_f (see §6 later). In the $t_f >> 1$ case, the root associated with the lowest $\widehat{\mathcal{L}}_0$ should prevail.

Let $u_\infty(t)$ denote the exact mathematical solution of the problem. It is well known that the initial-value problem of (1) using $u = u_\infty(t)$ may or may not recover the exact optimal solution $x_\infty(t)$ because (1) may be "open-loop" unstable. In general, a closed-loop control law, $u = u_\infty(x; t)$ is preferred. The goal of the present paper is *not* to find $u_\infty(t)$ and $x_\infty(t)$, but rather to find a stable "universal" closed-loop optimal control law without the need for detailed knowledge of $g(x)$.

In actual applications, the x needed in any closed-loop control law $u(x, t)$ is provided by sensor measurements. We assume both $x(t)$ *and* its time derivative, $\dot{x}(t)$, are directly measured, and denote the measured data by $x_*(t)$ and $\dot{x}_*(t)$, respectively. Since measurements always involve noise, we have:

$$x_*(t) = x(t) + O(\delta), \tag{10a}$$
$$\dot{x}_*(t) = \dot{x}(t) + O(\delta), \tag{10b}$$

where the $O(\delta)$ terms represent the (zero-mean) absolute error of the measurements. It is assumed that δ, the *error threshold* of the measurements, is small.

The main message of this paper is that if good quality $x_*(t)$ and $\dot{x}_*(t)$ data are provided, any scalar autonomous optimal control problem can be solved approximately if t_f is large and the value of $\widehat{\mathcal{L}}_0$, $\widehat{u}_0$ and $\widehat{x}_0$ are somehow provided. Detailed knowledge of $g(x)$ is not needed. Instead of a static closed-loop control law, a *dynamic* control law shall be proposed which can exploit both the availability of $\dot{x}_*(t)$ and the computational power of the controller to generate an approximate solution to the optimal control problem.

3 Known Analytical Results

It is well known that an exact static control law $u_\infty(x)$ can be solved for by quadrature—provided $g(x)$ is available [4, 5]. The classical approach defines the Hamiltonian by:

$$H(\dot{x}, x, u, \lambda) \equiv \lambda\frac{dx}{dt} - \mathcal{L}(u, x) \tag{11}$$

where λ is the so-called co-state variable. For the problem at hand, $\dot{x}$ in $H(\dot{x}, x, u, \lambda)$ can be eliminated using (1) to obtain:

$$\mathcal{H}(x, u, \lambda) \equiv H = \lambda\left(g(x) + bu\right) - \left(\frac{u^2}{2} + \phi(x)\right) \tag{12}$$

where $\mathcal{H}(x, u, \lambda)$ is an alternative representation of H in terms of x, u, and λ. Using calculus of variation, one obtains an algebraic equation for u

$$\left(\frac{\partial\mathcal{H}}{\partial u}\right)_{x,\lambda} = 0, \tag{13}$$

and an ODE for λ:

$$\frac{d\lambda}{dt} = -\lambda\frac{dg}{dx} + \frac{d\phi}{dx}. \tag{14}$$

For the specific (12) under consideration, (13) yields:

$$u = b\lambda. \tag{15}$$

Using (15) to eliminate λ from (14) in favor of u, one obtains:

$$\frac{du}{dt} = -u\frac{dg}{dx} + b\frac{d\phi}{dx}, \tag{16}$$

which, together with (1), poses a two-point boundary value problem for x and u.

A well known result of classical theory is that $\mathcal{H}(x, u, \lambda)$ is a constant when $\mathcal{H}$ is autonomous—*i.e.* when $\mathcal{H}$ has no explicit dependence on t. Evaluating the constant at the terminal point, one obtains:

$$\mathcal{H}(x, u, u/b) + \widehat{\mathcal{L}}_0 = 0, \tag{17}$$

which—for the problem at hand—is a quadratic equation for u. We denote the two roots by $u_\infty^\pm(x;\widehat{\mathcal{L}}_0)$:

$$u_\infty^\pm(x;\widehat{\mathcal{L}}_0) \equiv \frac{1}{b}\left(-g(x) \pm \sqrt{2b^2\left(\widehat{\mathcal{L}}(x)-\widehat{\mathcal{L}}_0\right)}\right). \quad (18)$$

where $\widehat{\mathcal{L}}(x)$, as defined by (6), depends on $g(x)$ and $\phi(x)$. By construction, they are potential exact optimal closed-loop *static* control laws:

$$u = u_\infty^\pm(x;\widehat{\mathcal{L}}_0). \quad (19)$$

Some useful properties of $u_\infty^\pm(x,\widehat{\mathcal{L}}_0)$ are:

$$u_\infty^+(x,\widehat{\mathcal{L}}_0) \;\geq\; u_\infty^-(x,\widehat{\mathcal{L}}_0), \quad (20a)$$

$$u_\infty^+(x,\widehat{\mathcal{L}}_0) + u_\infty^-(x,\widehat{\mathcal{L}}_0) \;=\; -\frac{2g(x)}{b}. \quad (20b)$$

Substituting (19) in (1), we obtain a single ODE for the *exact* optimal trajectory $x_\infty(t)$:

$$\frac{dx_\infty}{dt} = \pm\sqrt{2b^2\left(\widehat{\mathcal{L}}(x_\infty)-\widehat{\mathcal{L}}_0\right)}. \quad (21)$$

It is easily verified that the initial-value problem posed by (21) is always stable. The selection of the correct sign is intuitively obvious: the positive sign is to be used whenever $\widehat{x}_0 > x_o$; otherwise the negative sign prevails. It is easy to show that $u_\infty^+(x;\widehat{\mathcal{L}}_0)$ is associated with $\dot{x} > 0$, while $u_\infty^-(x;\widehat{\mathcal{L}}_0)$ is associated with $\dot{x} < 0$.

The exact optimum cost of the transition from x_o to $\widehat{x}_0$, to be denoted by $\Delta\mathcal{J}_\infty$, can be computed without knowing $x_\infty(t)$. Replacing the integrand $\mathcal{L}-\widehat{\mathcal{L}}_0$ in (8) using (11) and (17), we have, with the help of (18):

$$\Delta\mathcal{J}_\infty(x_o;\widehat{\mathcal{L}}_0) = \frac{1}{b}\int_o^\infty u_\infty^\pm \dot{x}_\infty dt$$

$$= \int_{x_o}^{\widehat{x}_0}\left(\frac{g(x)}{b^2} \pm \sqrt{\frac{2(\widehat{\mathcal{L}}(x)-\widehat{\mathcal{L}}_0)}{b^2}}\right) dx. \quad (22)$$

We shall find this formula useful later in evaluating the quality of our numerical simulations.

The weakness of the classical exact closed-loop *static* control law (19) is that detailed knowledge of $g(x)$ is needed—*i.e.* a validated mathematical model of the open-loop system dynamics is a prerequisite. The question being addressed is: if good quality $\dot{x}_*(t)$ is made available to the controller in addition to $x_*(t)$, can the need for detailed knowledge of $g(x)$ be reduced?

4 The New Approach

We now propose the following *dynamic control law* to be used by the controller:

$$\frac{du}{dt} = -\frac{2b}{\dot{x}_*(t)\Delta t}\left(H(\dot{x}_*(t), x_*(t), u, u/b) + \widehat{\mathcal{L}}_0\right) \quad (23)$$

where Δt is a judiciously-chosen "sufficiently small" positive time constant, and $H(\dot{x}, x, u, u/b)$ is the original representation of the Hamiltonian with λ replaced by u/b:

$$H(\dot{x}, x, u, u/b) = \frac{u\dot{x}}{b} - \left(\frac{u^2}{2} + \phi(x)\right). \quad (24)$$

When the physical system is "run," the Laws of Nature integrate (1)—which employs the *actual* $g(x)$—using $u(t)$ computed by the controller via numerical integration of (23) in real time. Note that the numerical evaluation of the right hand side of (23) needs only $u(t)$, $x_*(t)$, $\dot{x}_*(t)$, b, the value of $\widehat{\mathcal{L}}_0$, plus detailed knowledge of the user-specified function $\phi(x)$. All of the above information are assumed available to the controller. The value of Δt is the only "design parameter" in the control law—it should be chosen to be as small as pragmatically possible. To ensure stability of the numerical computations, Δt should be somewhat larger than t_s, the sampling time of the controller. The preferred numerical integration algorithm is the simple Euler (forward difference) method.

The properties of the coupled system of (1) and (23) can be studied by expressing the right hand side of (23) in terms of u and x. Replacing x_* and $\dot{x}_*$ in terms of the actual x and $\dot{x}$ using (10a,b), eliminating the actual $\dot{x}$ using (1), and taking advantage of the assumed smallness of the $O(\delta)$ terms, we obtain:

$$\frac{du}{dt} = -\frac{\left(u-u_\infty^+(x;\widehat{\mathcal{L}}_0)\right)\left(u-u_\infty^-(x;\widehat{\mathcal{L}}_0)\right)+O(\delta)}{\Delta t\left\{u-\left[u_\infty^+(x;\widehat{\mathcal{L}}_0+u_\infty^-(x;\widehat{\mathcal{L}}_0)\right]/2+O(\delta)\right\}}, \quad (25)$$

where the $u_\infty^\pm(x,\widehat{\mathcal{L}}_0)$'s were previously defined in (18). In the small Δt limit, the stability property of (25) can readily be examined by a variety of methods—*e.g.* by the graphical method of isoclines—under the assumption that the associated $x(t)$ is smooth and well-behaved. The rationale for putting $\dot{x}_*(t)$ in the denominator of (23) should be obvious. Both $u_\infty^\pm(x,\widehat{\mathcal{L}}_0)$'s can be shown to be stable "quasi-steady" solutions of (23) or (25). The initial condition $u(0)$ determines which quasi-steady solution will be selected.

When the measured data are noise-free (*i.e.* $\delta = 0$), it is easy to be convinced that $u(t)$ generated by (23) (using an appropriately selected $u(0)$ as initial condition) will rapidly approach either $u_\infty^+(x;\widehat{\mathcal{L}}_0)$ or $u_\infty^-(x;\widehat{\mathcal{L}}_0)$—with $O(\Delta t)$ error—after an initial transition layer of "thickness" $O(\Delta t)$.

The stability analysis of (25) also provides insights on the remaining issue: the choice of $u(0)$. The following conditions are readily arrived at:

$$bu(0)(\widehat{x}_0 - x_*(0)) > 0, \quad \dot{x}_*(0)(\widehat{x}_0 - x_*(0)) > 0. \quad (26)$$

In other words, the correct $u(0)$ must immediately steer $x_*(t)$ in the "right" direction. Any reasonable $u(0)$ satisfying (26) will do fine—$u(t)$ will rapidly approach the correct quasi-steady solution with a benign initial transient for $t = O(\Delta t)$, and follow it closely afterwards. Of course, if an approximate value of $u_\infty(0)$ were somehow known, its use not only would select the correct quasi-steady solution, but also would suppress the initial transient.

4.1 Effects of Measurement Noise

When measurement noise is present, the impacts of $\delta \neq 0$ on (23) can be studied by inspection of (25). The $O(\delta)$ term in the numerator of the right hand side is generally benign and harmless. However, the $O(\delta)$ term in the denominator is troublesome when u is in the vicinity of $\widehat{u}_0$ and $x(t)$ is in the vicinity of $\widehat{x}_0$. Hence, once $|x_*(t) - \widehat{x}_0|$ decays below δ, the dynamic control law (23) must be abandoned to avoid difficulties caused by the denominator. Instead of (23), a simple static control law can be used:

$$u = \widehat{u}_0 - \frac{K}{b}(x_*(t) - \widehat{x}_0) \tag{27}$$

where $K > 0$ is a judiciously chosen feedback gain. A simple way to chose the "right" value for K is as follows. While (23) is still in control, the current "effective" value of K defined by:

$$K_*(t) \equiv -\frac{b(u(t) - \widehat{u}_0)}{x_*(t) - \widehat{x}_0} \tag{28}$$

can be computed and stored in memory. As the value of $|x_*(t) - \widehat{x}_0|$ approaches $O(\delta)$, the value of $K_*(t)$ so computed is expected to approach a limiting value. When the static control law (27) takes over from (23), K can be chosen to be somewhat larger than the last computed value of K_* stored in the memory.

Any microprocessor-based programmable controller should have no difficulty switching between the dynamic (23) and the static (27) control laws as described above. While (27) is obviously a suboptimal control law, the penalty on $\mathcal{J}$ is $O(\delta)$ and should be of little concern for most practical problems.

5 Example

Consider a problem with $g(0) = 0$, $\phi(0) = 0$ and $d\phi/dx(0) = 0$. To be specific, we consider:

$$g(x) = \alpha x + x^2 G(x), \tag{29a}$$
$$\phi(x) = x^2 F(x). \tag{29b}$$

where α is an unknown constant of uncertain sign, $G(x)$ is some unknown differentiable function of x, and $F(x) > 0$ is a known (user-chosen) differentiable function of x. The example presented in Anderson and Kokotovic [3] is a special case of this problem with $b = 1$, $\alpha = 0$, $G(x) = x$, and $F(x) = 1$.

Substituting (29a) and (29b) in (6), we obtain $\widehat{\mathcal{L}}(x)$ for this problem:

$$\widehat{\mathcal{L}}(x) = x^2 \left\{ F(x) + [\alpha + xG(x)]^2 \right\} \tag{30}$$

The controller has no detailed knowledge of the system except for the following: the value of b, $\widehat{x}_0 = 0$, $\widehat{u}_0 = 0$ and $\widehat{\mathcal{L}}_0 = 0$.

When $|x_* - \widehat{x}_0| >> \delta$, the controller numerically integrates (23) which, for this specific example, is written out below:

$$\frac{du}{dt} = -\frac{2b}{\dot{x}_*(t)\Delta t} \left\{ \frac{u\dot{x}_*(t)}{b} - \frac{u^2}{2} - x_*^2(t) F(x_*(t)) \right\} \tag{31}$$

The initial condition $u(0)$ is required to satisfy (26). The value of Δt should be somewhat larger than t_s, the sampling time of the controller which is assumed to be "sufficiently small." In computer simulations, the relevant t_s is the numerical integration time step. Note that (31) needs no knowledge of α and $G(x)$. It does need good quality $x_*(t)$ and $\dot{x}_*(t)$ data, full knowledge of the user-chosen $F(x)$, and the numerical values of b.

When $|x_* - \widehat{x}_0|$ decays and becomes $O(\delta)$, the static control law (27) takes over (with $\widehat{u}_0 = 0$ and $\widehat{x}_0 = 0$). It is intuitively obvious that if $K > K_*$ is chosen, the theoretical stability requirement $K - \alpha > 0$ will be satisfied. The actual numerical value of K used has little theoretical or practical significance in the small δ limit.

Numerical simulations were performed using this hybrid control algorithm for a variety of α's, $G(x)$'s and $F(x)$'s. The sole design parameter Δt in (23) was chosen to be $2t_s$ where $t_s = 0.01$ was the numerical integration time step used in the simulation. The initial values for $u(0)$ were selected as discussed. Both $\dot{x}_*(t)$ and $x_*(t)$ were assumed independently measured, and measurement noises were emulated by $O(\delta)$ zero-mean random numbers ($\delta = 0$ and 0.05 were used), refreshed at every integration time step. The results were generally in full accordance with theoretical expectations. When $\delta = 0$ (measurement was noise-free) the dynamic (23) needed no assistance at all. When $\delta > 0$, the static (27) was needed, and was successful in controlling the final phase using $K = 1.5K_*$—the uncertainty of the sign and magnitude of α was handled in strides. The performance of the controller was generally excellent, as judged by comparing the numerical values of $\mathcal{J}$ obtained in the simulations with the theoretical $\mathcal{J}_\infty$ directly computed from (22)—so long as the noises contained in $x_*(t)$ *and* $\dot{x}_*(t)$ are *both* small. The interested readers can easily verify the above claims on desktop computers using their own favorite software packages.

6 Multiple Minimums

Consider now the case when the terminal condition (2b) is replaced by

$$x(t_f) = x_f, \quad t_f >> 1. \tag{32}$$

Without loss of generality, we assume that x_f is distinct from $\widehat{x}_0$.

When t_f is asymptotically large, the strategy is clear. The optimum trajectory now consists of three distinct phases: the first phase is a transition to go from x_o to $\widehat{x}_0$, the second phase is to stay put on $\widehat{x}_0$ until t_f is near, and the third phase is another transition to go from from $\widehat{x}_0$ to x_f. The first two phases are handled as described in the previous sections—use (23) for the transition follow by (27) for staying put. For the final phase, (23) is to be used again, initiated by a gentle "kick" in the right direction at the "right" time. The total cost is:

$$\mathcal{J} = t_f \widehat{\mathcal{L}}_0 + \Delta\mathcal{J}(x_0; \widehat{\mathcal{L}}_0) + \Delta\mathcal{J}(x_f; \widehat{\mathcal{L}}_0) \tag{33}$$

To arrive at $x = x_f$ precisely on time at $t = t_f$ is a very difficult task; the present strategy assumes that $O(1)$ error in t_f is acceptable.

We now assume that $\widehat{\mathcal{L}}(x)$ has multiple mininums, and denote their locations by $\widehat{x}_n$, $n = 0, 1, \ldots$. These minimums are ordered in ascending order of their associated values of $\widehat{\mathcal{L}}_n$:

$$\widehat{\mathcal{L}}_0 \leq \widehat{\mathcal{L}}_1 \leq \ldots. \tag{34}$$

When t_f is only moderately large, the magnitude of $t_f \widehat{\mathcal{L}}_0$ could be comparable to $\Delta\mathcal{J}(x_o, \widehat{\mathcal{L}}_0)$ and/or $\Delta\mathcal{J}(x_f, \widehat{\mathcal{L}}_0)$. When multiple minimums are present, competing strategic options need to be considered. It is now possible that some other $\widehat{x}_m$, not necessarily $m = 1$, can serve in the middle phase to achieve a lower total cost—when either or both x_o and x_f are very far from $\widehat{x}_0$. In order for the controller to assess the merits of such options and to make strategic tradeoffs, the transition costs $\Delta\mathcal{J}(x_o, \widehat{\mathcal{L}}_n)$'s and $\Delta\mathcal{J}(x_f, \widehat{\mathcal{L}}_n)$'s and the values of the associated $\widehat{u}_n$'s and $\widehat{x}_n$'s must somehow be made available.

7 Discussion

Obviously, if the measured data $x_*(t)$ and $\dot{x}_*(t)$ are either unreliable or of poor quality, the present controller—which has little knowledge of the system it is controlling—will understandably be unreliable and of poor quality. If the noise has zero mean and its frequency ω is "high," then a somewhat large value of Δt can be used to filter out the very high frequency components (*i.e.* $\omega\Delta t >> 1$). See Lam [6] for a more detailed discussion, including numerical results of simulations.

It is important to stress that $\dot{x}_*(t)$ should be directly measured and not obtained by numerical differentiation of $x_*(t)$. From the point of view of sensor technology, there is no reason to suppose that $\dot{x}_*(t)$ is fundamentally more difficult to measure than $x_*(t)$. In principle, the value of the system's b, which is needed by both (23) and (27), can also be directly measured by the controller by pulsing u.

The main weakness of the present theory is that it is applicable only to scalar and autonomous optimal control problems. It can be applied with trivial modifications to "nearly" autonomous scalar problems—provided the time dependence of the Hamiltonian is sufficiently weak, *i.e.*

$$\left(\frac{\partial\mathcal{H}}{\partial t}\right)_{x,u,\lambda} = O(\delta). \tag{35}$$

Note that (35) accepts for the possibility of $\mathcal{H}(t_1) - \mathcal{H}(t_2) = O(1)$ for $|t_1 - t_2| >> 1$, thus allowing a much richer set of strategic options when there are multiple minimums. The prospect of totally removing the scalar and the autonomous restrictions is not encouraging because the success of the present theory depends critically on the simplifications provided by both of these restrictions.

References

[1] Lam, S. H., "A Universal Robust Controller," DSC-Vol 61, in: Proc. ASME Dynamics System Division, G. Rizzoni (Ed.), Book No. H01105-1997, pp. 573-578, 1997.

[2] Lam, S. H., "A New Approach to Nonlinear Adaptive Control," *Journal of System Analysis, Modeling and Simulation*, Vol. **34**, No. 2, pp. 153-167. Gordon Breach, 1999.

[3] Anderson, B. D. O, and Kokotovic, P. V., "Optimal Control Problems Over Large Time Intervals," *Automatica*, **23**, No. 3, pp. 355-363, 1987.

[4] Bryson, A. E. jr., and Y. C. Ho, *Applied Optimal Control*, New York, Hemisphere, 1975.

[5] Lewis, F. L. and Syrmos, V. L., *Optimal Control*, Second Edition, John Wiley and Sons, Inc. 1995.

[6] Lam, S. H., "Universal Control Law and Rate Feedbacks," to be presented as paper #F-2e-02-4 at the 1999 IFAC World Congress, Beijing, China. July 5-9, 1999.