UNIVERSAL DYNAMIC CONTROL LAW AND RATE FEEDBACKS

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Abstract: A common approach to control problems is to look for a static control law through a detailed study of the known mathematical model of the system, assuming that the needed measurements of the state vector are available for closed-loop feedbacks. Recently, this author showed that a special *universal dynamic control law* can control a class of systems *without* the need of detailed knowledge of their mathematical models—provided good quality measured *rate* data of certain components of the state vector are available. The present paper shows that when additional knowledge beyond the minimum is available, the amount of needed rate data can be reduced. Numerical simulations are presented to elucidate the effects of measurement noises. *Copyright 1999 IFAC*

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1. INTRODUCTION

An often repeated conventional wisdom is that a control engineer must have an accurate and detailed mathematical model of the system to be controlled in order to be able to design an appropriate controller. In most cases, static control laws are used and full state feedbacks (with the help of estimators) are needed. Recently, Lam (1997a, 1997b, 1998) showed that for certain problems a *universal dynamic control law* (UDCL) is available which needs a very small amount of general information about the system—provided the feedbacks to the controller include certain *rate* data. The present paper examines the trade-offs between the need for detailed knowledge of the system model and the need for time derivatives of the outputs variables via an example. Simulation results are presented to elucidate the impact of finite measurement errors.

2. THE EXAMPLE PROBLEM

Consider the ODE system for $\mathbf{x}(t)$:

$$\frac{dx_1}{dt} = f_1(x_1, x_2, t), \tag{1a}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, x_3, t), \tag{1b}$$

$$\frac{dx_3}{dt} = f_3(x_1, x_2, x_3, t; u), \tag{1c}$$

where u is a scalar control "force." The functions f_1, f_2 and f_3 are assumed to be $O(1)$ smooth functions of their arguments, and that $\mathbf{x}$, u and t have been normalized so that they are all $O(1)$ entities. The goal is to find u—assuming whatever needed sensor measurements are available—such

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that the system state vector $\mathbf{x}$ is asymptotically stable about the origin. The desired "settling time" of the controlled system is $O(\tau)$ seconds.

This paper confines attention to the case when f_1, f_2 and f_3 have the following general properties:

$$\frac{\partial f_1}{\partial x_3} = 0, \quad (2a)$$
$$\frac{\partial f_1}{\partial x_2} \neq 0, \quad \text{(sign is known)}, \quad (2b)$$
$$\frac{\partial f_2}{\partial x_3} \neq 0, \quad \text{(sign is known)}, \quad (2c)$$
$$\frac{\partial f_3}{\partial u} \neq 0, \quad \text{(sign is known)}. \quad (2d)$$

The questions of interest are:

(1) what sensor measurements are needed as feedbacks when the controller's knowledge of the system is minimum?
(2) what benefits can be reaped if additional informations are available?

2.1 *Measurement Errors*

Since the controller is not fully informed about the system, it must totally trust the reliability and accuracy of the sensor measurements. In practice, measured data always contain noise. To distinguish measured data from the actual variables, all sensor measurements shall be marked by an asterisk superscript. Hence:

$$x_n^*(t) = x_n(t) + \delta_{n,0}, \quad (3a)$$
$$\dot{x}_n^*(t) = \dot{x}_n(t) + \delta_{n,1}, \quad (3b)$$
$$\ddot{x}_n^*(t) = \ddot{x}_n(t) + \delta_{n,2}, \quad (3c)$$
$$\cdots$$

where $\delta_{n,k}$, $\delta_{n,k}$, $\delta_{n,k}$, etc. represent the respective measurement noises—the second subscript represents the order of derivative involved. The UDCL, understandably, expects all measurement noises to be not only small but also to have zero mean values.

3. THE LEAST-INFORMED CONTROLLER

Let x_1 be the main output variable of interest. An alternative state vector $\mathbf{y}$ is introduced to replace $\mathbf{x}$, and its components y_1, y_2 and y_3 are defined by:

$$y_1 \equiv x_1, \quad (4a)$$
$$y_2 \equiv \frac{dy_1}{dt} + \frac{\Lambda_{11}}{\tau} y_1, \quad (4b)$$
$$y_3 \equiv \frac{dy_2}{dt} + \frac{\Lambda_{22}}{\tau} y_2, \quad (4c)$$

where Λ_{11} and Λ_{22} are $O(1)$ user-chosen positive numbers. The values of $\mathbf{y}(t)$ and its time derivatives computed from measured data shall also be marked by asterisk superscripts.

For this problem, Lam's *universal dynamic control law* (UDCL) is:

$$\frac{du}{dt} = -\frac{1}{B_{123}\Delta t}\left(\dot{y}_3^* + \frac{\Lambda_{33}}{\tau}(y_3^* - \varpi)\right), \quad (5)$$

where Δt is a "sufficiently small" positive time constant to be chosen, Λ_{33} is another user-chosen $O(1)$ positive number, and ϖ is yet another user-chosen function of t (called the *residual control* which will be exploited later), and

$$B_{123} \equiv \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial u} \quad (6)$$

which is assumed to be $O(1)$ and non-zero. Note that for the class of problems described by (1a,b,c), B_{123} is the only information about the system needed by the UDCL. In fact, since Δt is a free positive parameter to be chosen, only the sign of B_{123} is needed. In order to use (5), the task of the control engineer is reduced to choosing a sufficiently small Δt, pick Λ_{11}, Λ_{22} Λ_{33} and τ to obtain the desired transient properties, and pick $\varpi(t)$ for whatever purposes one may have in mind—provided reliable and accurate $y_3^*(t)$ *and* $\dot{y}_3^*(t)$ are available to be used for the evaluation of the right hand side of (5). Note that measurements of up to the third order time derivatives of $x_1(t)$ are needed to compute $\dot{y}_3^*(t)$.

In actual applications, Nature integrates (1a,b,c) using the real world as an analog computer, while the controller collaborates by numerical integration of (5) using a digital computer (Euler's method is preferred). Lam (1997a, 1997b, 1998) showed that (5) is stable in the small Δt limit for any reasonable initial condition $u(0)$ when f_3 depends on u linearly. The present paper's generalization that (5) remains stable even when the u dependence of f_3 is nonlinear is straightforward and can easily be verified by the readers. Whenever the computed $u(t)$ is finite and smooth, the associated bracketed term on the right hand side of (5) must be $O(\Delta t)$ when Δt is sufficiently small, *i.e.*:

$$\dot{y}_3^* + \frac{\Lambda_{33}}{\tau}(y_3^* - \varpi) = O(\Delta t). \quad (7)$$

Equation (7) is normally derived by applying the so-called *quasi-steady* approximation to (5), a technique well known in the theory of stiff ODEs (Lam, 1997a). Let $u_\infty(\mathbf{x}, t)$ denote the exact solution of (5) in the $\Delta t \to 0$ limit when all measurements are error-free (*i.e.* $\mathbf{y}^* = \mathbf{y}$). It can easily be shown that $u_\infty(\mathbf{x}, t)$ is identical to the exact solution to this problem obtained by the method of *feedback linearization* (Isidori,1995; Marino and Tomei, 1995). Unlike feedback linearization, which needs detailed knowledge of f_1, f_2 and f_3 in order to evaluate $u_\infty(\mathbf{x}; t)$, the UDCL as given by (5) needs only the sign of B_{123}—provided the properties listed in (2a,b,c,d) are satisfied. In other words, any smooth solution generated by (5)—with error-free measured data—is automatically an approximation to $u_\infty(\mathbf{x}; t)$:

$$u(t) = u_\infty(\mathbf{x}(t), t) + O(\Delta t). \tag{8}$$

The disadvantage is that measurements of up to the third order time derivatives of $x_1(t)$ are needed. It is of interest to note that the order of the highest time derivative needed by UDCL is called the *relative degree* in the nonlinear control literature. To show the effects of finite measurement errors, the quasi-steady approximation (7) is expressed in terms of the actual x_1 to yield:

$$\begin{aligned}\dddot{x}_1 &+ \frac{\Lambda_{11} + \Lambda_{22} + \Lambda_{33}}{\tau}\ddot{x}_1 \\ &+ \frac{\Lambda_{11}\Lambda_{22} + \Lambda_{22}\Lambda_{33} + \Lambda_{33}\Lambda_{11}}{\tau^2}\dot{x}_1 \\ &+ \frac{\Lambda_{11}\Lambda_{22}\Lambda_{33}}{\tau^3}x_1 - \frac{\Lambda_{33}\varpi}{\tau} + \delta = O(\Delta t), \end{aligned} \tag{9}$$

where δ represents the net consequence of all the (small) measurement errors incurred in approximating $y_3(t)$ and $\dot{y}_3(t)$ by $y_3^*(t)$ and $\dot{y}_3^*(t)$, respectively. Equation (9) represents the dynamics of $x_1(t)$ of the controlled system when $t >> \Delta t$. Note that as far as (9) is concerned, the impacts of δ and ϖ are indistinguishable. This observation shall be exploited in the simulations presented later.

Since ϖ is a forcing term in (9), it can be used by the controller to "steer" $x_1(t)$ to approximately track some user-specified desired trajectory for $t >> \tau$ —provided that the desired trajectory is sufficiently smooth.

4. EXPLOITING ADDITIONAL INFORMATION

When certain additional informations about f_1 and f_2 are known, significant benefits can be reaped.

4.1 *If $f_1(x_1, x_2, t)$ is precisely known*

If $f_1(x_1, x_2, t)$ is precisely known, (4a,b) yield:

$$y_1^* = x_1^*, \qquad y_2^* = f_1(x_1^*, x_2^*, t) + \frac{\Lambda_{11}x_1^*}{\tau}. \tag{10}$$

Equation (4c) yields:

$$y_3^* = \frac{D_1 f_1}{Dt} + \frac{\Lambda_{11} f_1}{\tau} + \frac{\partial f_1}{\partial x_2}\dot{x}_2^* + \frac{\Lambda_{22} y_2^*}{\tau}. \tag{11}$$

where

$$\frac{D_1}{Dt} \equiv \frac{\partial}{\partial t} + f_1(x_1^*, x_2^*, t)\frac{\partial}{\partial x_1^*}, \tag{12}$$

The recommended UDCL remains (5) without modification. By the use of (10) and (11), the bracketed term on the right hand side of (5) becomes:

$$\begin{aligned}&\frac{D_1}{Dt}(\frac{D_1 f_1}{Dt} + \frac{\Lambda_{11} f_1}{\tau} + \frac{\Lambda_{22} y_2^*}{\tau}) + \frac{D_1}{Dt}(\frac{\partial f_1}{\partial x_2^*})\dot{x}_2^* \\ &+\frac{\partial f_1}{\partial x_2^*}\ddot{x}_2^* + \frac{\Lambda_{33}}{\tau}y_3^* - \frac{\Lambda_{33}\varpi}{\tau},\end{aligned} \tag{13}$$

which the controller is capable of evaluation when $x_1^*(t)$, $x_2^*(t)$, $\dot{x}_2^*(t)$ and $\ddot{x}_2^*(t)$ are provided. In other words, the precise knowledge of $f_1(x_1, x_2, t)$ can remove the need of the third order time derivative, $\dddot{x}_1^*(t)$.

4.1.1. *If $f_2(x_1, x_2, x_3, t)$ is also precisely known*

If in addition $f_2(x_1, x_2, x_3, t)$ is also precisely known, it is easily shown that $\ddot{x}_2^*$ can be evaluated in terms of $x_1^*(t)$, $x_2^*(t)$, $x_3^*(t)$ and $\dot{x}_3^*(t)$.

For example, if $f_1 = \alpha x_2$ and $f_2 = \beta x_3$ (where α and β are constants), then we have $\dot{x}_1^*(t) = \alpha x_2^*(t)$, $\ddot{x}_1^*(t) = \alpha\beta x_3^*(t)$ and $\dddot{x}_1^*(t) = \alpha\beta\dot{x}_3^*(t)$.

4.2 *If $\frac{\partial f_1}{\partial x_1}$ is sufficiently negative*

If $\partial f_1/\partial x_1$ is known to be sufficiently negative, then x_1 is expected to be inherently stable without the assistance of u. Advantage can be taken of this observation to ignore x_1—provided the requirement that x_1 should asymptotically decay toward zero is relaxed.

For this case, x_2 replaces x_1 as the main output variable, and accurate measurements of $x_2^*(t)$ and its first and second order time derivative ($\dot{x}_2^*(t)$ and $\ddot{x}_2^*(t)$) are assumed available. The vector $\mathbf{y}$ is now defined by :

$$y_1 \equiv x_1, \tag{14a}$$
$$y_2 \equiv x_2, \tag{14b}$$
$$y_3 \equiv \frac{dy_2}{dt} + \frac{\Lambda_{22}}{\tau} y_2. \tag{14c}$$

Equation (5) remains the recommended dynamic control law except that B_{123} is now replaced by $B_{23} = (\partial f_2/\partial x_3)(\partial f_3/\partial u)$. The bracketed term on the right hand side of (5) is:

$$\ddot{x}_2^* + (\frac{\Lambda_{22}+\Lambda_{33}}{\tau})\dot{x}_2^* + \frac{\Lambda_{22}\Lambda_{33}}{\tau^2} x_2^* - \frac{\Lambda_{33}\varpi}{\tau} \tag{15}$$

In other words, the additional knowledge removes the need for measurement of the third order time derivative of $x_2(t)$ in the UCDL.

For this case, ϖ is allowed to depend on x_1^* and can serve as a residual control force to exert some influence on $x_1(t)$. For example, if it turns out that the magnitude of $\partial f_1/\partial x_1$ is not up to the task of guaranteeing asymptotic stability for x_1, the option of using the residual control $\varpi(x_1^*, t)$ can be explored (Lam, 1998).

4.3 *If both $\frac{\partial f_1}{\partial x_1}$ and $\frac{\partial f_2}{\partial x_2}$ are sufficiently negative*

If both $\partial f_1/\partial x_1$ and $\partial f_2/\partial x_2$ are known to be sufficiently negative, then x_1 and x_2 are expected to be inherently stable without the assistance of u. We can similarly take advantage of this observation.

For this case, x_3 is the sole variable of interest. The vector $\mathbf{y}$ is now identical to $\mathbf{x}$. The recommended dynamic control law remains to be (5) except that B_{123} is replaced by $B_3 = \partial f_3/\partial u$. The actual ODE to be numerically integrated by the controller is now simply:

$$\frac{du}{dt} = -\frac{1}{B_3 \Delta t}\left(\dot{x}_3^* + \frac{\Lambda_{33}}{\tau}(x_3^* - \varpi)\right). \tag{16}$$

In other words, the additional knowledge removes the need for the second and third order time derivatives.

For this case, ϖ is allowed to depend on x_1^* and x_2^* in addition to t. Similar comments at the end of the previous section are also applicable here.

5. THE ISSUE OF MEASUREMENT NOISE

Theoretically, the impact of noise (as represented by δ) is most felt by x_3 and u, less by x_2, and least by x_1. So long as all the needed measurements have good signal-to-noise ratios (*i.e.* $\delta << 1$), the UDCL will do just fine since their impacts are expected to be $O(\delta)$ (see (9)).

It is well known that numerical differentiation amplifies noise. Hence in general its use degrades the performance of UDCL. Consequently, the UDCL strongly advocates direct measurements of rate data. If for practical reasons direct measurements are not possible, then the option of using numerical differentiation could be considered. Some averaging and/or filtering of the data before and/or after numerical differentiations may be needed to modulate the increase of $\delta_{i,k}$ with increasing k. Note that (5) is essentially a low-pass filter for u. Hence, "high frequency" components of noise in δ can be somewhat muted by increasing the value of Δt—and in so doing sacrificing some "accuracy" of the resulting performance.

6. NUMERICAL SIMULATIONS

The following example is used to demonstrate the UDCL for the least-informed case:

$$f_1 = \alpha x_2 + g_1(x_1, t), \tag{17a}$$
$$f_2 = \beta x_3 + g_2(x_1, x_2, t), \tag{17b}$$
$$f_3 = g_3(x_1, x_2, x_3, t) + \gamma u. \tag{17c}$$

where α, β and γ are $O(1)$ constants or functions of t which are known not to change signs. Hence, $B_{123} = \alpha\beta\gamma \neq 0$. To simulate the controlled system, both (1a,b,c) and (5) are numerically integrated together on a desktop computer. It is assumed that x_1^* and $\dot{x}_1^*$ are directly measured, so that y_2^* can be computed via (4b) without the use of numerical differentiation. However, the higher time derivatives are computed by numerical differentiations. For example, $y_3^*(t)$ is computed via (4c) using the backward difference approximation for $\dot{y}_2^*(t)$:

$$y_3^*(t) = \frac{y_2^*(t) - y_2^*(t-\delta t)}{\delta t} + \frac{\Lambda_{22}}{\tau} y_2^*(t) \tag{18}$$

where δt is the numerical integration time-step used in the simulation. The needed $\dot{y}_3^*(t)$ on the right hand side of (5) is computed similarly—numerical differentiations is used twice in this evaluation.

According to (9), the measurement noise term δ can be emulated by using ϖ (instead of δ). Hence, in the simulations, no measurement noises are included in x_1^* and $\dot{x}_1^*$; the net effect of all measurement noises (and numerical differentiation errors) is emulated by choosing $\varpi(t) \neq 0$ (with $\delta = 0$ always).

Theoretically, the UDCL can handle any arbitrary differentiable g_1, g_2 and g_3, *deterministic or stochastic*. For concreteness, the following specific deterministic functions were chosen:

$$g_1 = x_1 - x_1^2 + x_1^3, \tag{19a}$$
$$g_2 = x_1 + 2x_2 + x_1 x_2, \tag{19b}$$
$$g_3 = x_1 - x_2 + 3x_3 + x_1 x_2 - 1. \tag{19c}$$

The initial conditions were $x_1(0) = -0.05$, $x_2(0) = -0.5$, $x_3(0) = -0.55$ and $u(0) = -2$. The integration stepsize was $\delta t = 0.01$, and Λ_{11}, Λ_{22}, and Λ_{33} were chosen to be 1.0, 1.0 and 3.0, respectively. The simulation results are presented in Figs. 1, 2, 3, 4 and 5. Fig. 1, which uses $\alpha = \beta = \gamma = 1$, $\Delta t = 2\delta t$ and $\tau = 0.5$, shows the ideal case when the measurements are noise-free ($\varpi = 0$)—even though numerical differentiations were used twice in evaluating the right hand side of the UDCL. Fig. 2 shows the effects of noise as emulated by $\varpi = 0.1\sin(100t)$—all other parameters are held fixed. Since $\Lambda_{33}/\tau = 6$, the amplitude of the emulated δ term is actually 0.6. The impacts of this noise on $x_1(t)$, $x_2(t)$ and even $x_3(t)$ are seen to be very minor indeed, but $u(t)$ now oscillates with a significant amplitude. Fig. 3 shows an identical run (as Fig. 2) except the magnitude of Δt is increased by a factor of three. It is seen that the amplitude of the undesirable oscillations is substantially reduced. Fig. 4 shows an identical run (as Fig. 3) except the value of τ is doubled. Fig.5 shows an identical run (as Fig. 4) except $\gamma = 1 + u^2$—in other words, $f_3(, x_1, x_2, x_3, t; u)$ now depends on u nonlinearly. The UDCL handles this complication in stride. When different $g_1(x_1, t)$, $g_2(x_1, x_2, t)$ and $g_3(x_1, x_2, x_3, t)$ (and different α, β, γ and initial conditions) were used than those shown in (19a,b,c), the simulated behaviors of the controlled system remained the same—only $u(t)$ was affected.

The UDCL can tolerate f_1 and f_2 to have a slight dependence on x_3 and u, respectively, and/or f_1 to have a *very* slight dependence on u—provided a lower bound for Δt is respected and the somewhat degraded resulting accuracy is considered acceptable. See (Lam, 1998) for a more detailed discourse on "nearly singular" problems.

The theory and the simulations performed above assumed that the actuator is capable of delivering any $u(t)$ computed by the UDCL. When the initial value for $u(0)$ is arbitrarily chosen, the UDCL needs some time to approach u_∞, and during this brief period the system could behave somewhat erratically. Experience shows that it is often a good idea to initially use a somewhat larger value of Δt and later reduce it to the desired sufficiently small value after the confusion is over. This was not done in the simulations presented here.

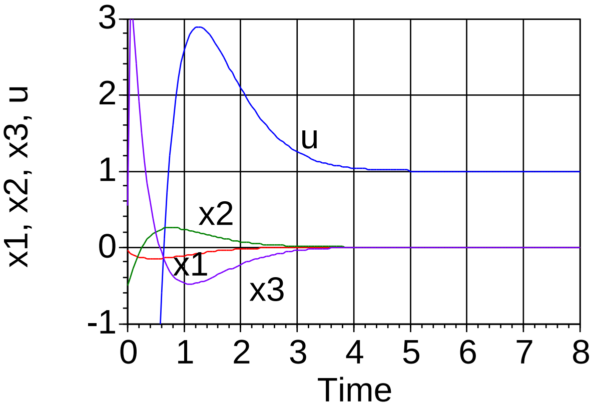

Fig. 1. x_1, x_2 x_3 and u vs. time with $\varpi = 0$ $\alpha = \beta = \gamma = 1$, $\Lambda_{11} = \Lambda_{22} = 1, \Lambda_{33} = 3$. $\delta t = 0.01$, $\Delta t = 2\delta t$, $\tau = 0.5$.

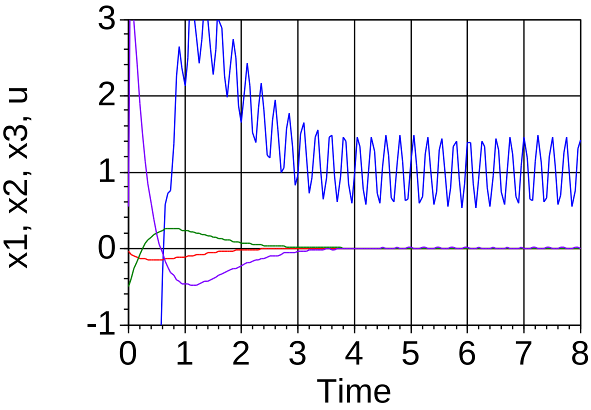

Fig. 2. x_1, x_2 x_3 and u vs. time with $\varpi = 0.1\sin(100t)$. All other parameters are identical to Fig. 1.

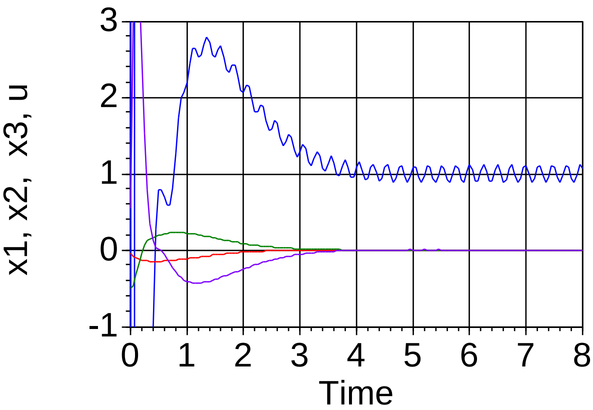

Fig. 3. x_1, x_2 and x_3 and u vs. time with $\varpi = 0.1\sin(100t)$. All other parameters are identical to Fig. 2 except $\Delta t = 6\delta t$.

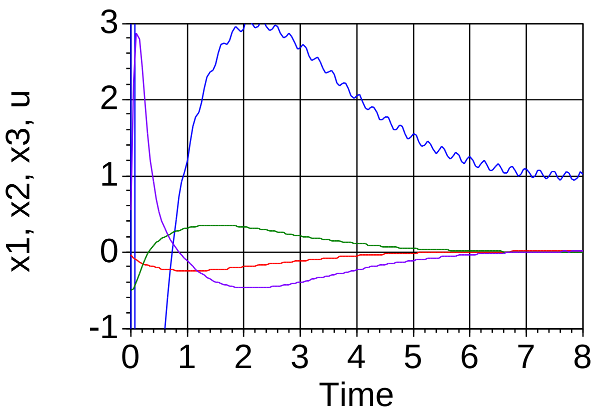

Fig. 4. x_1, x_2 and x_3 and u vs. time with $\varpi = 0.1\sin(100t)$. All other parameters are identical to Fig. 3 except $\tau = 1.0$.

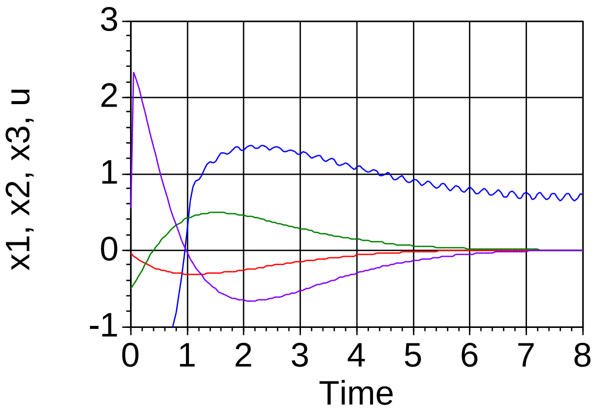

Fig. 5. x_1, x_2 and x_3 and u vs. time with $\varpi = 0.1\sin(100t)$. All other parameters are identical to Fig. 4 except $\gamma = 1 + u^2$.

7. CONCLUDING REMARKS

The main feature of the UDCL is that very little knowledge of the system to be controlled is needed. Consequently, real disturbances to the system or changes in the real system, whether deterministic or stochastic, are readily accommodated by the UDCL. The perceptive readers may have noticed that linearity of the system model is totally irrelevant from the point of view of the theory of UDCL. The basic ideas of UDCL can be applied to a general nonlinear system. In other words, the generic methodology can remain applicable even if conditions (2a,b,c,d) were not satisfied.

The information about the system most needed by UDCL is the so-called *pulse-response matrix*, represented by B_{123} and B_{23}, etc. in this paper. In the design of the system, which should include the design of the actuators and sensors in addition to the system itself, the designers should try to make the pulse-response matrix a constant matrix if at all possible. In the design of the sensors, direct measurements of rate data should be given high priority—making sure the needed highest time derivatives are measured reliably and accurately. From the vantage point of UDCL, the fabled successes of “inertial guidance” can be attributed to the measurement of accelerations in the control of spacecrafts and aircrafts. Once the pulse-response matrix is known and all needed rate data are guaranteed to have good signal-to-noise ratios, the UDCL does not need a mathematical model of the actual system at all.

Conceptually, both the UDCL and the feedback linearization approach use u to cancel out all the relevant actual terms and replace them with user-chosen terms. The UDCL approach uses the measured rate data to compute the relevant actual terms via the quasi-steady approximation (Lam, 1997b), while the conventional feedback linearization approach uses the detailed knowledge of the system. The price for the robustness of UDCL is the need for measured rate data with good signal-to-noise ratios, and some tolerance for finite tracking errors. Since detailed knowledge of the system is often uncertain, out-of-date, or perhaps even faulty, the option of using UDCL should be seriously considered whenever the needed sensor technology for good rate data measurements is available and high control accuracy is not a prime requirement.

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