# Quasi-Steady Approximation and Adaptive Nonlinear Controls*

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## ABSTRACT

Accurate and realistic mathematical modeling is always essential for conventional computer simulations such as CFD calculations of reacting flow systems. However, if the goal is not merely to make passive "what if" predictions but rather to use the available "actuators" in the system to exert active control over the dynamical behaviors of certain measured output variables, then the problem becomes a control problem. The present paper shows that (for finite dimensional dynamical systems only) it it possible to find "control laws" to accomplish the desired control objectives without having detailed knowledge of the mathematical model—provided reliable and accurate sensor measurements of the output variables are available.

## 1. INTRODUCTION

Consider the following general dynamical model of an engineering system [1, 2, 3, 4]:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f} + \sum_{i=1}^{I} \mathbf{b}_i u^i. \tag{1}$$

where $x$ is a $N$-dimensional column vector representing the state of the system. On the right hand side, the first term $\mathbf{f}(\mathbf{x}, t)$ represents the intrinsic (*i.e.* open-loop) dynamics of the system, and the second term represents the resultant effects of $I$ actuators which may be present in the system. Usually, $I$ sensors measurements are assumed available. In the community of computer simulations (such as CFD), the control term is nearly always absent. The goal of computations is usually a passive one: to obtain $\mathbf{x}(t)$ based on the best open-loop model $\mathbf{f}(\mathbf{x}, t)$ available for the system. In the control community, on the other hand, the control term is of prime importance. The goal is now an active one: to find "control laws" for the $I$ actuator control signals $u^i$'s such that the resulting sensor measurements will honor certain user-specified constraints. Consider the following scenario: the pressure of a combustion chamber is found to have undesirable oscillations when its fuel injection rate is held constant. Control engineers would look for a fuel injection control law using pressure feedback which could suppress the oscillations, while CFD'ers would most probably perform a number of computer simulations using different fuel injection time profiles to gain "understandings." A widely accepted conventional wisdom is that detailed and accurate knowledge of the open-loop term, $\mathbf{f}(\mathbf{x}, t)$, is important in either efforts. In the present paper, we shall show that it is possible to solve the control problem *without* detailed knowledge of $\mathbf{f}(x, t)$, provided the actuators and the sensors are intelligently chosen, and that the sensor measurements are reliable and have sufficiently good signal-to-noise ratios. The methodology used to achieve this feat is "quasi-steady approximation" [5, 6, 7] a concept familiar in reduced chemistry modeling of complex reaction systems.

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## 2. A ROBUST ADDAPTIVE CONTROLLER

We assume the system is to be controlled by a (microprocessor-based) controller which is responsible for computing and issuing the $I$ actuator control signals $u^i(t)$'s to the system using information provided by the $I$ sensor measurements. Each sensor signal $y^i$ is a measurement of the state of the system:

$$y^i = \Psi^i(\mathbf{x}, t), \qquad i = 1, \ldots, I, \tag{2}$$

where the $\Psi^i(\mathbf{x}, t)$'s are assumed differentiable with respect to its arguments. If $K = N - I > 0$, we can formally introduce $y^{I+k}$ by

$$y^{I+k} = \Psi^{I+k}(\mathbf{x}, t), \quad k = 1, \ldots, K, \tag{3}$$

so that $\mathbf{y} = \{y^1, \ldots, y^N\}$ is an alternative state vector of the system under consideration (the Jacobian of the coordinate transformation is assumed nonsingular). We call the first $I$ elements of $\mathbf{y}$ the *output variables*, and the remaining $K$ elements the *residual variables*.

### Dynamics of the Output Variables

Differentiating $y^i$ with respect to $t$, we obtain

$$\frac{dy^i}{dt} = g^i + \sum_{i'=1}^{I} B^i_{i'} u^{i'}. \tag{4}$$

where

$$g^i(\mathbf{x}, t) \equiv \frac{\partial \Psi^i}{\partial t} + \mathbf{c}^i \cdot \mathbf{f}, \tag{5}$$

$$\mathbf{c}^i(\mathbf{x}, t) \equiv \frac{\partial \Psi^i}{\partial \mathbf{x}}, \tag{6}$$

$$B^i_{i'} \equiv \mathbf{c}^i \cdot \mathbf{b}_{i'}. \tag{7}$$

Since the $\Psi^i$'s are known to be independent of the $u^i$'s, the $g^i$'s are known to be independent of time derivatives of the $u^i$'s.

In the following developments, we shall assume that $B^i_{i'}(t)$ is known since, according to (4), it can in principle be directly measured on-the-fly by the controller by pulsing the $u^i$'s.

### ODE-Based Constraints

In a control problem, the behaviors of the measured $y^i(t)$'s are of special interest to the control engineers. For example, it may be desirable (for tracking or regulation problems) to keep the $y^i$'s below some user-specified accuracy threshold $\delta$. Instead of such *algebraic constraints*, we propose to impose the following ODE-based constraints:

$$\Phi^i(\dot{y}^i, \phi^i, t) \equiv \frac{dy^i}{dt} + \phi^i(\mathbf{y}, t) = O(\delta), \tag{8}$$

where the $\phi(\mathbf{y}, t)$'s are $O(1)$ functions of the arguments and can be freely chosen by the users. For example, a possible choice is:

$$\phi^i(\mathbf{y}, t) = \frac{1}{\tau} \sum_{i'=1}^{I} \Lambda^i_{i'} \left( y^{i'} - \varpi^{i'} \right) \tag{9}$$

where $\tau > 0$ is a time constant, $\Lambda^i_{i'}$ is a real matrix and the *residual controls* $\varpi^i(t)$'s are allowed to depend on $t$ and perhaps the residual variables $y^{I+k}$'s. For example, if we choose $\Lambda^i_{i'}$ to be positive-definite and the $\varpi^i$'s to be zeros, then this special ODE-constraint guarantees that the $y^i$'s are $O(\delta)$ for $t >> \tau$.

**Exact Static Control Law $u^i_\infty(\mathbf{x},t)$**

The control law $u^i = u^i_\infty(\mathbf{x},t)$ required to precisely honor the ODE-based constraint (8) with $\delta = 0$ can readily be found as follows. Eliminating $dy^i/dt$ between (4) and (8) with $\delta = 0$, we obtain:

$$\sum_{i'=1}^{I} B^i_{i'} u^{i'}_\infty = -(\phi^i + g^i). \tag{10}$$

Assuming $B^i_{i'}$ to be nonsingular, we can solve for $u^i_\infty$ directly to obtain:

$$u^i_\infty(\mathbf{x},t) = -\sum_{i'=1}^{I} [B^{i'}_i]^{-1}(\phi^i + g^i). \tag{11}$$

The use of this static control law $u^i = u^i_\infty(\mathbf{x},t)$ as given by (11) requires detailed knowledge of the system model (specifically, the $g^i$'s), and full state feedback, *i.e.* $\mathbf{x}(t)$ itself.

**The Proposed Dynamic Universal Controller When $B^i_{i'}$ is Nonsingular**

The present paper proposes the following dynamic universal control law:

$$\frac{du^i}{dt} = -\frac{1}{\Delta t}\sum_{i'=1}^{I} W^i_{i'}(u^{i'} - u^{i'}_\infty). \tag{12}$$

where $W^i_{i'}$ is a positive-definite real matrix, and $\Delta t$ is a small time constant—both are to be user-chosen. Note that $u^i_\infty(\mathbf{x},t)$ as given by (11) has no dependence on the $u^i$'s or their time derivatives. In the $\Delta t << 1$ limit, the quasi-steady approximation can be applied, yielding for $t >> \Delta t$:

$$u^i = u^i_\infty(\mathbf{x},t) + O(\Delta t). \tag{13}$$

Hence, if we choose $\Delta t = O(\delta)$, numerical integration of (12) will recover approximately the static control law. At this point, the use of (12) offers no advantage at all over using $u^i = u^i_\infty(\mathbf{x},t)$ directly—other than the introduction of constraint errors.

However, (12) does offer an outstanding advantage which only becomes obvious when it is rewritten in an alternative form. By definition of the matrix inverse, we have

$$\sum_{i"=1}^{I} [B^{i"}_i]^{-1} B^{i"}_{i'} = \delta^i_{i'} \tag{14}$$

where $\delta^i_{i'}$ is the identify matrix. We now rewrite $W^i_{i'}$ as follows:

$$W^i_{i'} = \sum_{i"=1}^{I} W^i_{i"}\delta^{i"}_{i'}. \tag{15}$$

Using (15) in (12) and with the help of (14), (10) and (4), we obtain,

$$\frac{du^i}{dt} = -\frac{1}{\Delta t}\sum_{i'=1}^{I} Z^i_{i'}(\frac{dy^{i'}}{dt} + \phi^{i'}), \tag{16}$$

where $Z^i_{i'}$ is a nonsingular matrix:

$$Z^i_{i'} \equiv \sum_{i"=1}^{I} W^i_{i"}[B^{i'}_{I"}]^{-1}. \tag{17}$$

Since (16) is mathematically identical to (12) which is known to be stable in the small $\Delta t$ limit, so is (16). Unlike (12), however, (16) does *not* require knowledge of $u^i_\infty(\mathbf{x}, t)$. Instead, (16) requires $B^i_{i'}(t)$, the $\dot{y}^i(t)$'s and whatever $y^n(i)$'s are needed for the evaluation of the user-chosen $\phi^i(\mathbf{y}, t)$'s, in addition to specific (but rather flexible) choices for $W^i_{i'}$ and $\Delta t$. For example, the choices $W^i_{i'} = \delta^i_{i'}$ and $\Delta t = \delta$ would do very well indeed. Certainly, the (microprocessor-based) controller is capable of numerically integrating this ODE, leaving the "real" dynamical equations of the system (1) to be integrated by the Laws of Nature. It is readily verified that together they will honor the ODE-based constraints—no explicit knowledge of $\mathbf{f}(\mathbf{x}, t)$ is needed.

## 3. WHEN $B^i_{i'}$ IS SINGULAR OR NEARLY SINGULAR

Many engineering systems have singular or nearly-singular $B^i_{i'}$'s. Generalizations to include such problems involves the extensive use of the concepts and tools of singular value decomposition. The singular value decomposition [8] of $B^i_{i'}$ is written as follows:

$$B^i_{i'} = \sum_{i'''=1}^{I} [U^{i'''}_i]^T \omega(i''') V^{i'''}_{i'} \tag{18}$$

where $U^{i'''}_i$ and $V^{i'''}_{i'}$ are unitary matrices and $\omega(i''') \geq 0$ are the singular values ordered in descending magnitudes. The number of non-zero singular values is by definition the traditional rank of $B^i_{i'}$. Consider now the case when $J > 0$ of the smallest singular values of $B^i_{i'}$ are $O(\epsilon)$ where $\epsilon$ is either precisely zero or is a "sufficiently small" number. The *epsilon-rank* of $B^i_{i'}$ is defined as the number of above threshold singular values. *i.e.* $I - J$. Because $U^i_{i'}$ is a unitary matrix, we have:

$$\sum_{i=1}^{I} U^{I+1-j}_i B^i_{i'} = \omega(I+1-j) V^{I+1-j}_{i'} = O(\epsilon), \qquad j = 1, \ldots, J. \tag{19}$$

The $U^{I+1-j}_i$'s, which are the bottom $J$ rows of $U^i_{i'}$ and play a major of role in the subsequent development, are called the *left epsilon-vectors* of $B^i_{i'}$.

In order for the control problem to have a solution, $J$ additional *derived output variables* $y^{I+j}$'s must be found, and the resulting extended $M \times I$ rectangular matrix $B^m_i$ (where $M = I + J$) must have full epsilon-rank. For the sake of simplicity, we limit our theoretical developments to the following special ODE-based constraints for the $M$ output variables:

$$\Phi^m = \frac{dy^m}{dt} + \frac{1}{\tau} \sum_{m'=1}^{M} \Omega^m_{m'} (y^{m'} - \varpi^{m'}) = O(\delta), \qquad m = 1, \ldots, M. \tag{20}$$

where $\tau$, $\Omega^m_{m'}$ and the $\varpi^m$ are parameters analogously defined as previously in (9). Unlike the previous case, however, there are certain restrictions on $\Omega^m_{m'}$ and the $\varpi^m$'s as we shall see presently.

Only the theoretical results for the case $J = 1$ (and thus $M = I + 1$) are presented here. For details of the derivations, see [5]. The sole newly promoted $y^M$ is:

$$y^M = \Psi^M(\mathbf{x}, t; \epsilon) \equiv \sum_{i=1}^{I} U^I_i \left( \frac{dy^i}{dt} + \frac{1}{\tau} \sum_{i'=1}^{M} \Omega^i_{i'} y^{i'} \right) \tag{21}$$

which can be rewritten in the following more elegant form:

$$\sum_{i=1}^{I} U^I_i \Phi^i = 0, \tag{22}$$

provided

$$\Omega^i_M = -\tau[U^I_i]^T, \qquad i = 1, \dots I, \tag{23}$$

and that the $\varpi^i$'s satisfy

$$\sum_{i,i'=1}^{I} U^I_i \Omega^i_{i'} \varpi^{i'} = 0. \tag{24}$$

Except for (23), all remaining elements of $\Omega^i_{i'}$ are completely free to be chosen. Note that $y^M(t)$ is expressed completely in terms of the $I$ primary sensor signals $y^i$'s. Note that whenever $\epsilon \neq 0$, $\Psi^M(\mathbf{x}, t; \epsilon)$ has an $O(\epsilon)$ dependence on the $u^i$'s.

Differentiating (21) with respect to time, the ODE for $y^M$ can be derived with the help of (1). We assume that the resulting $M \times I$ real matrix $B^m_i(t)$ has full epsilon rank and is known to the controller. Its singular value decomposition can be computed. The epsilon-inverse of $B^m_i$ is denoted by $[B^m_i]^+$ and is given by:

$$[B^m_i]^+ \equiv \sum_{i'=1}^{I} [\check{V}^{i'}_i]^T \frac{1}{\check{\omega}(i')} \check{U}^{i'}_m. \tag{25}$$

where $\check{U}^i_m$, $\check{V}^i_{i'}$ and $\check{\omega}(i) > \epsilon$ are the corresponding unitary matrices and singular values of $B^m_i$. If we pad the row vector $U^I_i$ (the left epsilon-vector of $B^i_{i'}$) with a trailing zero, the resulting row vector (to be denoted by $\tilde{U}^M_m$) is automatically qualified to be a left epsilon-vector of $B^m_i$, and is expected to be a good approximation to the true left null-vector of $B^m_i$, $\check{U}^M_m$, since $\epsilon << 1$.

The proposed universal dynamic control law is:

$$\frac{du^i}{dt} = -\frac{1}{\Delta t} \sum_{m=1}^{M} \check{Z}^i_m \left( \frac{dy^m}{dt} + \frac{1}{\tau} \sum_{m'=1}^{M} \Omega^m_{m'} (y^{m'} - \varpi^{m'}) \right) \tag{26}$$

where

$$\check{Z}^i_m \equiv \sum_{i'=1}^{I} W^i_{i'} [B^m_{i'}]^+ \tag{27}$$

and $\Delta t > 0$ and $W^i_{i'}$ have the same physical meanings as before.

If the original $B^i_{i'}$ is strictly singular (*i.e.* $\epsilon = 0$), it is straightforward to show that in the limit of $\Delta t \to 0$ the quasi-approximation can be applied to (26), and the dynamics of the resulting output variables will indeed honor the ODE-based constraints for $t >> \Delta t$. However, if the original $B^i_{i'}$ is nearly-singular (*i.e.* $\epsilon \neq 0$), then depending on the sign of $\epsilon$, a lower limit may need to be imposed on the value of $\Delta t$. This is because when $dy^M/dt$ on the right hand side of (26) is eliminated by using its own dynamical equation, the $g^M$ term contains an $O(\epsilon)$ contribution involving $du^i/dt$. Stability consideration of (26) then requires that $\Delta t >> |\epsilon|$ whenever the sign of $\epsilon$ is uncertain. Pragmatically speaking, the controller is most likely to use Euler's method to numerically integrate (26). Consequently, a lower bound for the accuracy threshold for ODE-constraints must always be respected whenever $\epsilon$ is not precisely zero—when the original $B^i_{i'}$ is nearly-singular.

## 5. DISCUSSIONS

The present theory requires that reliable sensor signals with sufficiently good signal-to-noise ratios are available. It strongly prefers to directly measure the time derivatives of the output variables instead of getting them by numerical differentiation. In addition, the impulse-response matrix $B^m_i(t)$ is assumed available. If the choice and design of the actuators are at the disposal of the control engineers, a time-invariant and nonsingular $B^i_{i'}$ would be preferred. The option of direct measurements of $B^m_i(t)$ on-the-fly by pulsing the $u^i$'s is assumed available.

## 4. AN EXAMPLE

Consider the $N = 3$, $I = 1$ case with

$$\mathbf{f} = \begin{vmatrix} f_1(x_1, x_2, x_3, t) \\ f_2(x_1, x_2, x_3, t) \\ f_3(x_1, x_2, x_3, t) \end{vmatrix}, \qquad \mathbf{b}_1 = \begin{vmatrix} \epsilon \\ 1 \\ 0 \end{vmatrix}, \qquad y_1 = x_1. \tag{28}$$

All superscripts have been moved to become subscripts to avoid confusion with exponents. We assume that $\partial f_1/\partial x_2$ is of $O(1)$ and is never zero, while $\epsilon$ is a small parameter of uncertain sign. We have $B_{1,1} = \epsilon$ and the problem is either singular or nearly-singular. For the ODE-based constraints, we choose, in accordance to (23),

$$\Omega_{m,m'} = \tau \begin{vmatrix} 1/\tau_1 & -1 \\ 0 & 1/\tau_2 \end{vmatrix} \tag{29}$$

where $\tau_1$ and $\tau_2$ are positive free parameters. We require $\varpi_1 = 0$ in accordance to (24). The new derived output variable is simply $y_2 = dy_1/dt + y_1/\tau_1$. The dynamical equation for $y_2$ can readily be derived and expressed in the form of (4). We have $g_2 = \partial f_1/\partial t + \mathbf{f}_1 \partial f_1/\partial x_1 + \mathbf{f}_2 \partial f_1/\partial x_2 + \mathbf{f}_3 \partial f_1/\partial x_3 + \epsilon du_1/dt$, and the new entry to the impulse-response matrix is $B_{2,1}(t) = \partial f_1/\partial x_2 + \epsilon(\partial f_1/\partial x_1 + 1/\tau_1)$. The universal dynamic control law is:

$$\frac{du_1}{dt} = -\frac{1}{\Delta t}\left(\frac{B_{2,1}}{\epsilon^2 + B_{2,1}^2}\right)\left(\frac{dy_2}{dt} + \frac{1}{\tau_2}(y_2 - \varpi_2)\right). \tag{30}$$

where $\varpi_2(x_3, t)$ is available to exert some influence on $x_3(t)$.

If $\epsilon = 0$ precisely, (30) can accept asymptotically small values for $\Delta t$ and remain stable. However, if $\epsilon \neq 0$, then the $\epsilon du_1/dt$ term implicit in $dy_2/dt$ through $g_2$ can compete with the $du_1/dt$ term on the left hand side. If $\epsilon$ is negative, a lower bound $\Delta t > |\epsilon|$ must be imposed. The readers can easily verify using desktop computers that (30) indeed solves the control problem provided $B_{2,1}(t)$ is $O(1)$ and is provided to the controller, and that the lower bound $\Delta t > |\epsilon|$ is respected.

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