Wake Thrust of Heat Addition in the Flow Field

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Abstract

What happens in the far wake when heat is added (or removed) from a small finite volume in a steady, uniform compressible flow?

1 Formulation

Consider a steady flow problem with upstream velocity $\mathbf{U}_o$ pointing in the $+x$ direction. We enclose the flow field of interest by a large control volume V, denote the surface of this control volume by S and its outward unit normal by $\mathbf{n}$. The integral form of the continuity equation is:

$$\int\int_S \rho(\mathbf{q}\cdot\mathbf{n})dA = 0 \tag{1}$$

The integral form of the (inviscid) momentum equation is:

$$\int\int_S \rho\mathbf{q}(\mathbf{q}\cdot\mathbf{n})dA = \mathbf{F} - \int\int_S p\mathbf{n}dA. \tag{2}$$

where $\mathbf{F}$ is any "external force" acting on the control volume and p is the perturbation (gage) pressure. Multiplying (1) by $\mathbf{U}_o$ and subtracting the result from (2), we obtain:

$$\int\int_S \rho(\mathbf{q}-\mathbf{U}_o)(\mathbf{q}\cdot\mathbf{n})dA = \mathbf{F} - \int\int_S p\mathbf{n}dA. \tag{3}$$

The x component of this vector equation is:

$$\int\int_S \rho(q_x - U_o)(\mathbf{q}\cdot\mathbf{n})dA = F_x - \mathbf{e}_x \cdot \int\int_S p\mathbf{n}dA. \qquad (4)$$

Now we choose the control volume to be a large cylinder of radius R, with its "ends" at $X = -L$ abd $x = +L$. We further denote:

$$q_x = U_o + u', \quad q_y = v', \quad q_z = w' \qquad (5)$$

and introduce cylindrical coordinates (r, θ) to replace y and z:

$$r = \sqrt{y^2 + z^2}, \quad \varpi = \sqrt{v'^2 + w'^2}, \qquad (6)$$

where ϖ is the fluid velocity component in the r direction. Equation (4) can now be written as:

$$\int\int_{\text{cylindrical S}} \rho u' \varpi R d\theta dx + \int\int_{\text{rear S}} \rho(U_o + u')u' dydz = F_x. \qquad (7)$$

The pressure integral on the side cylindrical surface has no component in the x direction, and the integral on the ends (upstream and rear) surfaces vanish in the large L limit since $p \to 0$ there.

Note that if $F_x > 0$, it is usually called "thrust," while otherwise it is usually called "drag." If the disturbance is simply heat addition (or removal) in a finite volume of the flow field (without the benefit of a physical device such as an engine), then $F_x = 0$. Thus we have:

$$\int\int_{\text{cylindrical S}} \rho u' \varpi R d\theta dx + \int\int_{\text{rear S}} \rho(U_o + u')u' dydz = 0. \qquad (8)$$

2 The Subsonic Case

In the subsonic case, it is easy to show that all the flow disturbance (u' and ϖ) are expected to decay with R faster than $R^{-/12}$ (using subsonic linear potential theory). Thus in the large R limit the cylindrical surface integral is zero. Equation (8) then concludes that u' far downstream of the perturbation must be zero. Hence, we obtain the simple conclusion that in subsonic flow the streamtube which experienced the heat addition does not change the value of its velocity far downstream; obviously the fluid in this streamtube far downstream is hotter and has lower density than the freestream values.

3 The Supersonic Case

In the supersonic case, heat addition generates a wave system, and the first term of (8) will be nonzero and can be shown to be always negative (use linear supersonic potential theory; see next paragraph and my *Notes on Classical Sonic Boom Theory*), and its magnitude is called the "wave drag." However, since the sum is zero, equation (8) says the second term must also be nonzero, and must be positive. In other words, the consequence of heat addition is that a hot jet can be found in the wake, and the "wake thrust" of this hot jet precisely balances the wave drag emitted away from the heat addition region.

For a uniform steady supersonic flow in the $+x$ direction, the Mach cone in the x, r plane is a line pointing to the north-east. The perturbation velocity vector is perpendicular to the Mach cone. For a compressive wave, the perturbation pressure is positive, and from Bernoulli's equation u' is negative. The requirement that the perturbation velocity vector be perpendicular to the Mach cone says that ϖ must be positive (away from the x axis). Hence the product $u'\varpi$ is negative. For an expansive wave, u' is positive. Using the same geometric argument, ϖ must then be negative (toward the x axis). Hence, the product $u'\varpi$ is again negative.

Heat addition in a finite volume in the flow field is expected to generate a compressive wave, while heat removal is expected to generate an expansive wave.

4 Remarks on Heat Removal

The exposition above is valid far heat addition or removal. It is of interest to note that for supersonic flows the wave drag term is always negative, whether the disturbance is caused by heat addition or removal. Hence, the "wake thrust" is always positive, hence u' of the streamtube in question is positive in the wake. Intuitively, this makes sense for the heat addition case. The theory here says u' there is also positive for the heat removal case—which is counter-intuitive. It is helpful to recall that the effect of heat addition to a one-dimensional streamtube is to drive the flow velocity toward from the local sonic velocity, while heat removal does the opposite. Thus, right after the heat addition or removal, u' is negative and positive, respectively. It thus appears that for the heat addition case u' goes from negative to positive, while for the cooling case u' simply stays always positive.