

MAE 222
Mechanics of Fluids
Princeton University

Assignment # 7

April 1, 1998

Due on Wednesday, 2PM, April 8, 1998

Chapter 6. Differential Analysis of Fluid Flow. This chapter uses quite a bit of standard mathematics. Please let me know if any of the mathematics used here needs additional clarifications.

§6.1, Fluid Element Kinematics. In this section, we are introduced to four mathematical operations, each one has a clear physical meaning. In §6.1.1, we see the *substantial derivative* in (6.6), again. You must know what it means physically. Then we see the *gradient operator* in (6.7). (we met it before on page 32 in hydrostatics. What does a gradient vector mean physically?). Then we see the *divergence operator* in (6.9), and its physical meaning is given by (6.9). Then we see the *curl operator* in Cartesian coordinates presented between (6.16) and (6.17) on page 212. The curl of the velocity vector field \mathbf{V} is *VORTICITY*, denoted by ζ :

$$\zeta \equiv \nabla \times \mathbf{V}. \quad (1)$$

The *average angular velocity* of a tiny glob of fluid is denoted by ω . The physical meaning of vorticity is: it is simply twice the average angular velocity!!!

Given a scalar field and/or a vector field, you are expected to be able to “operate” on the given field by these operators (at least in Cartesian coordinates), and know physically what the results mean.

§6.2. Conservation of Mass. The concept of conservation of mass is eloquently expressed in finite control volume formulation by (6.19). If we pick Cartesian coordinates, then we can derive (6.27) from (6.19)! If density ρ is a constant following a tiny glob of fluids,

then the continuity equation is (6.30) in vector form and (6.31) in long hand Cartesian coordinate form. What happens if you want to do things in cylindrical polar coordinates? The vector form is still (6.30), but the long hand polar coordinate form becomes (6.33). What happens in spherical coordinates? To get the long hand spherical coordinate form, you have to either derive it or look it up!

For two-dimensional, incompressible (ρ does not change as you follow a tiny glob of fluid) flow, the concept of streamfunction is a useful one. The equation $\psi(x, y) = \text{constant}$ gives you a curve, and the curve is a streamline! The same “trick” works for a flow with cylindrical symmetry (see (6.42)).

§6.3. Conservation of Linear Momentum. The punch line here is the mess in (6.50a,b,c) if we agree to use Cartesian coordinates. If you multiply each equation by $\delta x \delta y \delta z$, the volume of a tiny cube of fluid, then the right hand side is just mass times acceleration! The first term on the left hand side is the contribution of the gravitational (volumne) force. The other three terms are the contributions of the surface forces. What are the surface forces? Well, we have the pressure contribution which is (by definition of pressure) normal to the surface. Then we have surface forces due to viscosity which may point in any direction (at this time). At this time, we have not committed ourselves on how to represent these surface forces yet. The full discussion of the viscous forces is given later in §6.8.1 and §6.8.2.

§7.4. Inviscid Flow . If we just keep the pressure contribution, and neglect all the viscous contributions, we would arrive at the *inviscid* momentum equation given by (6.51a,b,c). The vector form is given by (6.52), and is much prettier! (Inviscid assumption is justified by the Reynolds number of the flow being large).

The Bernoulli's equation is now rederived in §6.4.2. The answer is given by (6.56). If the fluid density is a constant, we get (6.57). See the four bullets just below (6.58)!

In §6.4.3-6.4.4, YMO talks about *irrotational flows*. You will later see a movie on vorticity. I will also supplement with some lecture materials in class on vorticity. Generally speaking, vorticity is a

big deal as far as airplanes are concerned!

Here is the main “theorem” (commonly known as Kelvin’s theorem). If a glob of constant density fluid once upon a time had no vorticity (no average angular velocity), and viscosity surface forces are negligible (the Reynolds number is large), then this glob will continue to have no vorticity. So, when we have a large Reynolds number incompressible flow which has a uniform velocity profile far, far upstream flowing over an airfoil, we expect the flow field to be irrotational. If so, the velocity field can be written in the form of (6.65), where ϕ satisfies (6.66), the Laplace equation. All aircrafts before and during world war II were designed with (6.66)—with a very subtle twist which will be presented in class. The main concept to note is: Laplace Equation is linear, and the principle of superposition applies!

For irrotational flow, the Bernoulli constant is a constant over the whole flow field—we are no longer restricted to using the Bernoulli’s equation along a streamline!!!

§6.5-6.6-6.7. Potential Flows. With the help of a set of “elementary solutions,” we can construct interesting looking flow fields. I will talk about the issue of DRAG in class—and the main reason why the airplane is such an efficient machine.

§6.8 Viscous Flow. Now, we need to relate the surface forces due to viscosity to the flow field. Clearly, if there is no motion, there is no viscous surface force. Clearly, if the fluid moves as a solid body, there is no viscous surface force. Hence, the viscous surface force must depend on the spatial derivatives of the velocity vector, somehow. The formulation is due to Navier and Stokes, and the stress-deformation relation is given by (6.118a,b,c,d,e,f) for Cartesian coordinates, and by (6.119a,b,c,d,e,f) for cylindrical polar coordinates. The corresponding linear momentum equations are given by (6.120a,b,c) and (6.121a,b,c) in §6.8.2.

§6.9-6.10 Some Simple Solutions, and Other Aspects. When the pipe flow Reynolds number is below about 2300 (see Moody Chart on page 335), we have the laminar regime. §6.9.3 show how the Navier-Stokes equation can be solved to give a theoretical prediction for laminar flows. The problems of transition to turbulence

and turbulent flow itself are still unsolved problems and are current research topics.

Problems at end of Chapter 6, page 269:

- Problem 6.1.
- Problem 6.2.
- Problem 6.3.
- Problem 6.5.
- Problem 6.6.
- Problem 6.19. Remember, the Bernoulli's constant is a constant over the irrotational flow field! And: given the velocity potential, you find the velocity by taking the gradient of the velocity potential!
- Problem 6.46 The top plate is fixed, and the bottom plate is moving with horizontal velocity U . The pressure gradient $\partial p/\partial x$ is given. And you are encouraged to assume $\partial/\partial x = 0$ (the flow at any x-station looks the same). You should be find the ordinary differential equation for the x-component of the velocity, and integrate it to find the solution analytically.

Use email or the newsgroup to ask questions.