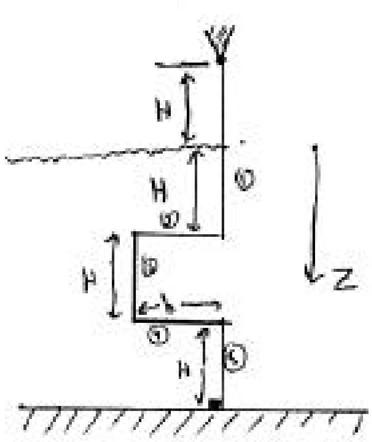
Homework #3 - Solutions

(2.43) A rigid gate of width w is hinged without friction at a point H above the surface. Find the ratio b/H at which the gate is about to open



Assume: 1) weight of gate is negligible
2) negligible friction in hinge
3) ρ = constant (incompressible)
4) g = constant
5) $P_{\text{above water}}$ = P to right of gate = P_a

Gate opens when the moment about the hinge becomes positive (ie counterclockwise) so the critical point is when $\sum$ moments = 0

$\tilde{M}_o = \hat{s} \times \hat{F}$ or

(F = hydrostatic pressure $= \rho g z$
$s = \perp$ distance from hinge)

$M_o = \int_{z_1}^{z_2} \rho g(H+z) z\,dz$ (vertical piece)

$M_o = \int_0^b \rho g(H+z) s\,ds$ (horizontal piece)

$= \frac{\rho g}{2}(H+z)b^2$

section ①: $\rho g\left(\frac{Hz^2}{2} - \frac{z^3}{3}\right)_0^H = \left[\frac{1}{2}H^3 + \frac{1}{3}H^3\right]\rho g$

section ③: $\rho g\left(\frac{Hz^2}{2} - \frac{z^3}{3}\right)_H^{2H} = \left[2H^3 + \frac{8}{3}H^3 - \frac{1}{2}H^3 - \frac{1}{3}H^3\right]\rho g$

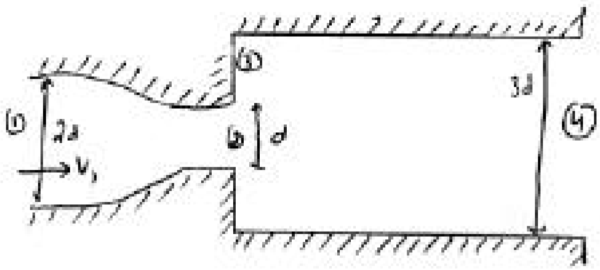
section ⑤: $\rho g\left(\frac{Hz^2}{2} - \frac{z^3}{3}\right)_{2H}^{3H} = \left[\frac{9}{2}H^3 + 9H^3 - 2H^3 - \frac{8}{3}H^3\right]\rho g$

section ② $\frac{\rho g}{2}(H)b^2 \quad = \frac{1}{2}\rho g b^2 H$

section ④ $-\frac{\rho g}{2}(2H)b^2 \quad = -\rho g b^2 H$ (add

$\cancel{\rho g}\frac{27}{\cancel{2}}H^{\cancel{3}\,2} - \frac{1}{\cancel{2}}\cancel{\rho g}b^2\cancel{H} = 0 \quad \Rightarrow 27H^2 = b^2$ or $\boxed{\frac{b}{H} = \sqrt{27} = 5.2}$

(4.6) For the given duct, pressure at the exit is atmospheric, ρ = const, duct has constant width w. a) Sketch flow pattern b) Can Bernoulli's equation be used from 2 → 4? c) Is $P_2 = P_3$? d) Find P_{2g} in terms of ρ and V_1. e) Find P_{1g} in terms of ρ and V_1.

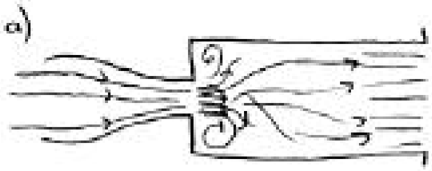


P_a

Assumptions:

1) ρ = const
2) viscous force on fluid << pressure force
3) flow is steady (in the mean)
4) Uniform (1D) velocity profiles at (1), (2), (4)
5) $P_4 = P_a$
6)

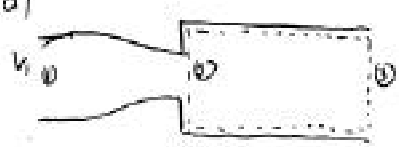
a)



b) No, Bernoulli's can't be used from (2) → 4 because the turbulence ~~#~~ in the large chambers means there are viscous losses in the flow and streamlines don't exist from (2) → (4)

c) If P_3 was significantly different from P_2, the streamlines at (2) wouldn't be parallel to each other; they would be "bent" by the pressure difference so it would be eliminated.

d)



Continuity: $\dot{m}$ = const ⇒ $\dot{m}_1 = \dot{m}_2 = \dot{m}_3$

$$\rho V_1 A_1 = \rho V_2 A_2 = \rho V_3 A_3$$

$$\rho w 2d V_1 = \rho w d V_2 = \rho w d 3 V_3$$

$$\Rightarrow V_2 = 2V_1 \, , \; V_3 = \frac{2}{3} V_1$$

Momentum: ΣF = net outflow of momentum

$$P_2 A_2 - P_3 A_3 = -\dot{m} V_2 + \dot{m} V_3$$

$$P_{2g} \cdot 3d = \rho V_1 2d \left(\frac{2}{3} V_1 - 2V_1\right) \Rightarrow \boxed{P_{2g} = -\frac{8}{9} \rho V_1^2}$$

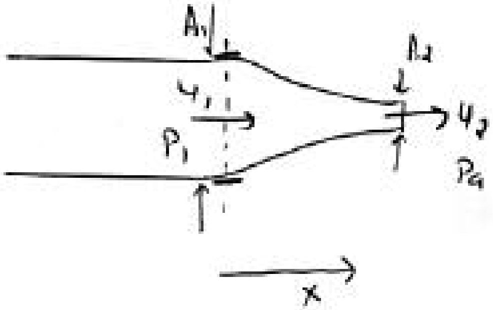
e) use Bernoulli's from (1) to (2)

$$\frac{P_1}{\rho} + \frac{1}{2} V_1^2 = \frac{P_2}{\rho} + \frac{1}{2} V_2^2 = \frac{P_a}{\rho} - \frac{8}{9} \rho V_1^2 + \frac{1}{2} V_2^2$$

$$P_{1g} = \rho \left[\frac{1}{2} V_2^2 - \frac{8}{9} V_1^2 - \frac{1}{2} V_1^2\right] = \rho \left[2V_1^2 - \frac{8}{9} V_1^2 - \frac{1}{2} V_1^2\right] \Rightarrow$$

$$\boxed{P_{1g} = \frac{11}{18} \rho V_1^2}$$

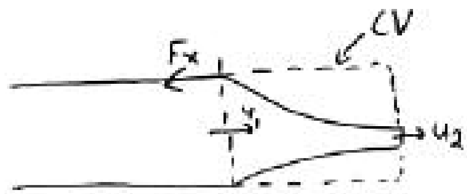
(4.12) Incompressible, 1D flow exits steadily into the ambient atmosphere from the contraction which is bolted onto a constant area duct. Area ratio $A_1/A_2 = 4$. If the total force on the bolts is F_x, find $F_x/\rho u_1^2 A$. Would the analysis hold if flow direction was reversed



Assume: 1) ρ = const
2) steady
3) 1-D flow
4) viscous forces can be neglected
5) $P_2 = P_a$

From momentum equation:

ΣF = net outflux of momentum



in x-direction:

$$-F_x + P_1 A_1 - P_2 A_1 = -\rho A_1 u_1^2 + \rho A_2 u_2^2$$

From continuity:

$$\dot{m} = \text{const} = \cancel{\rho} u_1 A_1 = \cancel{\rho} u_2 A_2 \implies u_2 = u_1 \frac{A_1}{A_2} = 4u_1$$

From Bernoulli (all 4 conditions ok)

$$\frac{P_1}{\rho} + \frac{1}{2}u_1^2 = \frac{P_2}{\rho} + \frac{1}{2}u_2^2 = \frac{P_2}{\rho} + \frac{1}{2}(4u_1)^2$$

$$P_2 = P_a \qquad P_1 = P_a + \frac{15}{2}u_1^2\rho$$

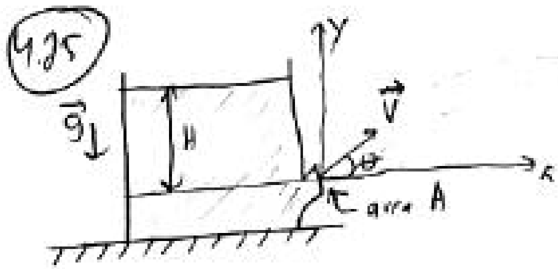
$$\therefore \quad -F_x + \left[P_a + \frac{15}{2}\rho u_1^2 - P_a\right]A_1 = \rho A_1\left[-u_1^2 + \frac{A_2}{A_1}u_2^2\right] = \rho A_1\left[\frac{1}{4}(4u_1)^2 - u_1^2\right]$$

$$-F_x = -\frac{15}{2}\rho u_1^2 A + 3\rho u_1^2 A_1$$

$$\boxed{\frac{F_x}{\rho u_1^2 A_1} = \frac{9}{2}}$$

If flow is reversed P_2 must be $< P_a$ (to "suck" air into the duct). In this case, the streamlines are different at 2 due to the unequal pressures at that point
$\implies$ new problem.

4.25



Jet issues smoothly from hole.
Find max jet height (depending on H and θ).
Find horizontal force exerted on tank.
Ignore losses

Assumptions: 1) no losses (no viscosity)
2) tank is so big that $H \approx$ constant (quasi-steady)
3) $P_{jet} = P_{atm}$
4) 1D flow in jet
5) ~~[illegible]~~ width of jet is small (ie neglect height differences in vertical jet cross-section)

Apply Bernoulli from surface to jet exit:

$$\frac{P_{atm}}{\rho} + \cancel{\frac{1}{2}V_s^2}^{\,0} + gH = \frac{P_{atm}}{\rho} + \frac{1}{2}V_j^2 + g(\emptyset)$$

$$\text{or } \frac{1}{2}V_j^2 = gH \quad \Rightarrow V_j = \sqrt{2gH}$$

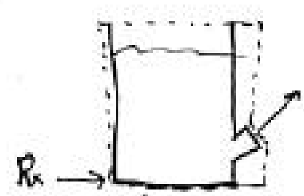
Apply Bernoulli from jet exit to max jet height:

$$\cancel{\frac{P_{atm}}{\rho}} + \frac{1}{2}V_j^2 + g(\emptyset) = \cancel{\frac{P_{atm}}{\rho}} + \frac{1}{2}V_{h_{max}}^2 + gh_{max}$$

V has 2 components: $V_x = V\cos\theta \quad V_y = V\sin\theta$ ($V_y = 0$ at max height, V_x = constant in jet)

$$\cancel{\frac{1}{2}V_{jx}^2} + \frac{1}{2}V_{jy}^2 = \underbrace{\cancel{\frac{1}{2}V_{jx}^2}}_{=\frac{1}{2}(V_{h_{max}})_x^2} + \frac{1}{2}(0) + gh_{max} \quad \Rightarrow h_{max} = \frac{1}{2g}V_{jy}^2 = \boxed{H\sin^2\theta = h_{max}}$$

Use x-mom eqn:



ΣF_x = net x-mom. outflow

$\Sigma F_x = \int \rho V_x (\vec{V}\cdot\vec{n})dA$ ← notice I drew CV so $\vec{n}\cdot\vec{V} = V_j$

$\Sigma F_x = \rho V_x V_j A = \rho V_j^2 \cos\theta A$

$\Sigma F_x = R_x$ (pressure is equal at both left and right boundary)
(R_x is force required to hold tank in place – by friction at lower surface maybe)

$$\boxed{R_x = \rho V_j^2 \cos\theta A} \boxed{= 2\rho g H A \cos\theta}$$