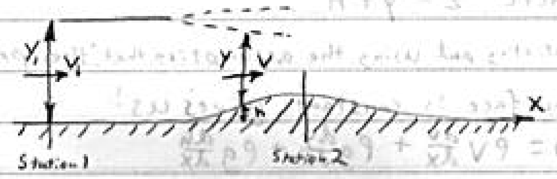
MAE224 Hmwk #4 Solutions

11.9



A fluid in an open channel flow encounters a bump. a) Derive an expression for the differential change in the height of the fluid $\left(\frac{dy}{dx}\right)$. b) Discuss the implications of this result.

Assumptions: The flow is steady, incompressible, inviscid and one-dimensional. The pressure along the top surface is atmospheric. Also, the channel has uniform width (w)

For steady, incompressible flow, the mass conservation equation can be written as:

$$y_1 V_1 = y V \quad (\text{recall } w = \text{const})$$

where y is the fluid height and V the fluid velocity at any point.

Differentiating yeilds:

$$0 = y\frac{dV}{dx} + V\frac{dy}{dx}$$

$$\Rightarrow \frac{dV}{dx} = \frac{-V}{y}\frac{dy}{dx}$$

Next we write Bernoulli's Equation for a streamline along the surface of the fluid:

(cont...)

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V^2 + \rho g z$$

where $z = y + h$

Differentiating and using the assumption that the pressure along the surface is constant gives us:

$$0 = \rho V \frac{dV}{dx} + \rho g \frac{dy}{dx} + \rho g \frac{dh}{dx}$$

Substituting from above:

$$0 = \rho V \left(\frac{-V}{y}\right)\frac{dy}{dx} + \rho g \frac{dy}{dx} + \rho g \frac{dh}{dx}$$

Divide by ρg and group terms:

$$0 = \left(\frac{-V^2}{gy} + 1\right)\frac{dy}{dx} + \frac{dh}{dx}$$

Recall $F^2 = \frac{V^2}{gy}$

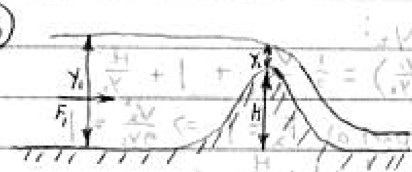
$$\boxed{\text{a)}\ \frac{dy}{dx} = \frac{1}{F^2 - 1}\frac{dh}{dx}}$$

b) If the flow is subcritical everywhere, then $\frac{dy}{dx}$ is negative for positive $\frac{dh}{dx}$:

If the flow is supercritical everywhere, then $\frac{dy}{dx}$ is positive for positive $\frac{dh}{dx}$:

If $\frac{dy}{dx}$ is finite, then $F = 1$ can only occur at a point where $\frac{dh}{dx} = 0$, which is station 2. This allows for flow to switch between subcritical and supercritical.

(11.13)



$y_1 \gg y_2$
$Z = y_1 - H$

Water in a two-dimensional channel flows over a submerged obstacle as shown. a) Provide arguments to justify that $F_2 = 1$. b) Find y_2 as a function of Z. c) Explain what would happen if there is a tall verticle wall on the right side.

Assumptions: The flow is steady, incompressible, inviscid and 1-D. The pressure along the top surface is atmospheric, and the width of the channel is constant. Also, $y_1 \gg y_2$.

a) The formula derived in 11.9 also applies to this problem:

$$\frac{dy}{dx} = \frac{1}{F^2-1}\frac{dh}{dx}$$

From the figure, we can see that $\frac{dy}{dx}$ at the peak is not zero. Since $\frac{dh}{dx}$ is zero at the peak, F_2 must be equal to 1.

For part b), we can start with mass conservation:

$$y_1 V_1 = y_2 V_2 \qquad (\text{steady, incompressible, } w = \text{const})$$

And we also apply Bernoulli's equation along a streamline on the surface:

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g y_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g (y_2 + H)$$

Using mass conservation, we can substitute in for V_1:

$$\tfrac{1}{2}\rho\left(\frac{y_2 V_2}{y_1}\right)^2 + \rho g y_1 = \tfrac{1}{2}\rho V_2^2 + \rho g (y_2 + H)$$

(cont...)

(11.13)

now divide by $\rho g y_2$:

$$\frac{1}{2}\left(\frac{y_2^2}{y_1^2}\right)\frac{V_2^2}{g y_2} + \left(\frac{y_1}{y_2}\right) = \frac{1}{2}\frac{V_2^2}{g y_2} + 1 + \frac{H}{y_2}$$

Okay, recall that from part a) $F_2 = 1 \Rightarrow \frac{V_2^2}{g y_2} = 1$

$$\frac{1}{2}\left(\frac{y_2^2}{y_1^2}\right) + \left(\frac{y_1}{y_2}\right) = \frac{1}{2} + 1 + \frac{H}{y_2}$$

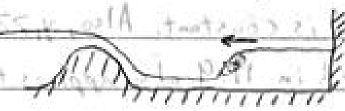
Using the assumption that $y_2 << y_1 \Rightarrow \left(\frac{y_2}{y_1}\right)^2 \approx 0$

$$\frac{y_1}{y_2} = \frac{3}{2} + \frac{H}{y_2} \Rightarrow y_1 = \frac{3}{2} y_2 + H$$

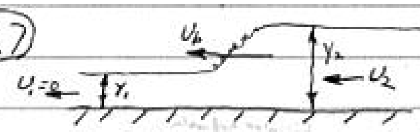
recall $z = y_1 - H$:

$\boxed{b)\ z = \frac{3}{2} y_2}$ $\quad y_2 = \frac{2}{3} z$

c) If there was a wall on the right hand side, a hydraulic jump would propagate upstream as shown below:



(11.27)



A bore is moving with a velocity U_b into a stagnant tidal channel of depth Y_1, as shown above. When $U_b=\sqrt{3gY_1}$, find U_2/U_b

Assumptions: The flow is incompressible, inviscid and 1-D. As usual, the pressure along the top surface is atmospheric, and the channel width is constant. By transforming to a coordinate system which moves with the hydraulic jump, we can make the problem steady:

For steady, incompressible flow in a constant-width channel, mass conservation yeilds:

$$U_b Y_1 = (U_b - U_2) Y_2$$

$$Y_1 = \left(1 - \frac{U_2}{U_b}\right) Y_2$$

$$\Rightarrow \frac{U_2}{U_b} = 1 - \frac{Y_1}{Y_2}$$

From Smits, Eq. (11.8):

$$\frac{Y_2}{Y_1} = \frac{1}{2}\left(\sqrt{1+8F_1^2} - 1\right) \quad \text{where } F_1 = \frac{U_b}{\sqrt{gY_1}}$$

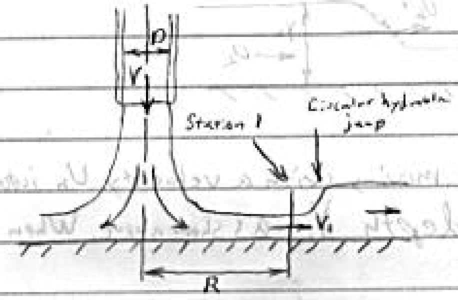
$$\frac{Y_2}{Y_1} = \frac{1}{2}\left(\sqrt{1+8\frac{U_b^2}{gY_1}} - 1\right)$$

$$\Rightarrow \frac{U_2}{U_b} = 1 - \frac{2}{\sqrt{1+8\frac{U_b^2}{gY_1}} - 1} \qquad U_b = \sqrt{3gY_1} \Rightarrow F_1 = \sqrt{3}$$

$$\frac{U_2}{U_b} = 1 - \frac{2}{\sqrt{1+(8)(3)} - 1}$$

$$\boxed{\frac{U_2}{U_b} = \frac{1}{2}}$$

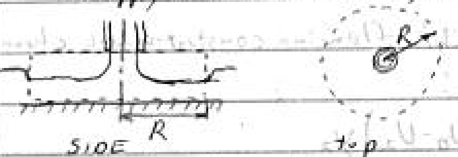
11.33



A circular jet impinges on a flat plate. The exit velocity (V) is 10 m/s and the exit diameter (D) is 10 mm. Also, $R = 8D$ and $Y_1 = \frac{D}{4}$. Find (a) the Froude number at 1, and (b) the depth after the hydraulic jump.

Assumptions: The flow is steady, incompressible, inviscid and axisymmetric. The velocity is uniform at any station and the jump is very thin. The pressure is atmospheric along the surface.

To apply conservation of mass, draw a cylindrical control volume:



$$\rho A V = \rho_1 A_1 V_1$$

$$\pi \left(\frac{D}{2}\right)^2 V = (2)\pi R\, y_1 V_1 \quad \text{recall, } y_1 = \frac{D}{4},\ R = 8D$$

$$\left(\frac{D}{2}\right)^2 V = (2)(8D)\left(\frac{D}{4}\right) V_1 \Rightarrow V_1 = \frac{1}{16} V$$

$$F_1 = \frac{V_1}{\sqrt{g y_1}} = \frac{V}{16\sqrt{g(\frac{1}{4})D}} \Rightarrow F_1 = \frac{10}{16\sqrt{9.8(0.0025)}} \qquad \boxed{\text{a) } F_1 = 4}$$

Smits Eq. (11.8): $\frac{y_2}{y_1} = \frac{1}{2}\left(\sqrt{1+8F_1^2} - 1\right)$

$$\Rightarrow \boxed{\text{b) } y_2 = 12.9\text{ mm}}$$

The jump can be assumed thin if R is large.