MAE 224 Notes #1b
Elements of Thermodynamics and Fluid Mechanics

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February 3, 1999

1 Supplemental Notes

Consider any problem with N variables and parameters (including both inputs and outputs). We assume that there are 3 basic dimensions involved, M (mass), L (length) and T (time).

We denote the N variables and parameters by x_n. The dimension of each x_n is expressed as follows:

$$\text{dim of } [x_n] = (M)^{\alpha_n}(L)^{\beta_n}(T)^{\gamma_n}, \quad n = 1, \ldots, N. \tag{1}$$

The matrix of dimensions for this problem is readily constructed.:

$$\begin{array}{ccccc} & x_1 & x_2 & \ldots & x_N \\ \text{M} & \alpha_1 & \alpha_2 & \ldots & \alpha_N \\ \text{L} & \beta_1 & \beta_2 & \ldots & \beta_N \\ \text{T} & \gamma_1 & \gamma_2 & \ldots & \gamma_N \end{array}. \tag{2}$$

Now consider the dimensionless parameter Π_i defined by:

$$\Pi_i = (x_1)^{\kappa_{1i}}(x_2)^{\kappa_{2i}} \ldots (x_N)^{\kappa_{Ni}} \tag{3}$$

where the κ_{ni}'s are unknowns. Since Π_i is dimensionless, we have:

$$\alpha_1\kappa_{1i} + \alpha_2\kappa_{2i} + \ldots + \alpha_N\kappa_{Ni} = 0 \tag{4}$$

$$\beta_1\kappa_{1i} + \beta_2\kappa_{2i} + \ldots + \beta_N\kappa_{Ni} = 0 \tag{5}$$

$$\gamma_1\kappa_{1i} + \gamma_2\kappa_{2i} + \ldots + \gamma_N\kappa_{Ni} = 0. \tag{6}$$

We now have a system of 3 linear algebraic equations for the N unknown κ_{ni}'s—the ith-subscript is being held fixed for each Π_i under investigation.

To obtain a set of κ_{ni} who honors this set of algebriac equations, we can proceed simple-mindedly as follows.

- Set $i = 1$, pick $\kappa_{41} = 1$ and $\kappa_{51} = \ldots = \kappa_{N1} = 0$, and solve the resulting equations for κ_{11}, κ_{21} and κ_{31}. When this is done, a usable Π_1 has been found!

- Now set $i = 2$, pick $\kappa_{52} = 1$ and $\kappa_{42} = \kappa_{62} = \ldots = \kappa_{N2} = 0$, and solve the resulting equations for κ_{12}, κ_{22} and κ_{32}. When this is done, a usable Π_2 has been found!

- Continuing, a total of $N - 3$ usable Π_i will be found.

It is moderately easy to see that these Π_i's are independent. So, apparently, there are $N-3$ independent Π_i's for every problem! But the latter conclusion is not generally correct!

The procedure as described fails if the relevant square 3×3 matrix involved in the solving of the three remaining κ's is singular. So the procedure must be modified to deal with the general case. That is the role of the Buckingham's Π Theorem:

> The number of independent Π_i's is $N - R$ where R is the rank of the matrix of dimensions.

Note: the maximum possible value for R is 3 for problems in mechanics, but it may be less than 3. So, the number of independent Π_i's is at least $N - 3$, but may be more. How do you conclude, with conviction, what the number is for your problem? You need to find the rank of your matrix of dimensions.

In practice, one usually do not use the above bulleted procedure, which is moderately messy, to find the Π_i's. Watch how I do it in class.

1.1 Temperature

Temperature, denoted by θ in Professor Smits' notes, is formally a new basic dimension introduced by thermodynamics. However, with a little bit of cleverness, we can eliminate it from our considerations, and stay with our three M L T basic dimensions of mechanics.

Intuitively, temperature is associated with energy in some vague way. So, if we use something that has the dimension of energy (per unit mass) to represent temperature, we can achieve our goal.

There are many ways to do this. For example, the internal energy of a fluid (in fact, any material) is a monotonic function of temperature. For problems with limited range of temperature variation in the neigbourhood of θ_o, the following formula is usually pretty good:

$$e = e_o + C_v(\theta - \theta_o), \tag{7}$$

where e is internal energy per unit mass (in units of velocity-square), e_o and θ_o are reference values, and C_v, the specific heat at constant volume, is assumed to be a constant. Instead of θ, use $C_v\theta$ as your temperature variable.

If you are dealing with a perfect gas, the equation of state is then:

$$p = \rho\mathcal{R}\theta \tag{8}$$

where $\mathcal{R}$ is the gas constant (universal gas constant divided by the molecular/atomic weight). It provides another way of eliminating θ out of your hair: use $\mathcal{R}\theta$ as your temperature variable.