MAE 224 Notes #3a
Elements of Thermodynamics and Fluid Mechanics

S. H. Lam

February 15, 1999

1 Reading and Homework Assignments

The problems are due on Wednesday, February 24, 1999, 5PM. Please submit your homework to the MAE 224 homework **IN** tray outside D-302, E.Q.

- Read Chapter 2, (Fluid Statics; skip §2.10, §2.11), Chapter 3, (Basic Principles I), Chapter 4, (Basic Principles II), Chapter 5, (Equations of Motion in Integral Form) in Professor Smits' *A Physical Introduction to Fluid Mechanics.*

  It is my understanding that Chapters 3 and 5 were covered, including homeworks, in MAE 223.

1. Chapter 2, Fluid Statics. Study §2.1 and feel comfortable and familiar with (2.3), (2.4) and (2.5) on page 47. Once you know the fluid is at rest, you know the presure distribution in the fluid: it depends only on the altitude of the point of interest. If density is a constant (such as water), the pressure changes linearly with altitude (the pressure decreases with altitude). §2.3-2.8 are expositions on how to compute forces and moments acting on various kind of surfaces in contact with the fluid. §2.9 is the Archemedes Principle. I will show in class how to use it to solve certain problems treated previously in a more elegant way. Skip §2.10 and §2.11.

- Do problem 2.43 on page 99. See Fig. 2.43. You may assume the atmospheric pressure to be a constant (the gage pressure of the atmosphere is zero). It is obvious that if b is too short, this frictionless hinged gate will open clockwise. Hint: find the moment about the hinge due to the gage pressure of the water. The critical length of b is when this moment is zero.

2. Read Chapter 3 and remind yourself what you had learned in MAE 223.

3. The big deal in Chapter 4 is the Bernoulli's equation (4.1) on page 133, AND the four conditions for its validity just below on the same page. Remember, the "flow is steady" statement not only excludes honest-to-goodness unsteady flows, but also "steady" turbulent flows (when your streamline of interest goes through a turbulent flow region); the "there is no friction" statement is normally supported by the observation that the Reynolds number of the problem is large, and our interests lie outside (laminar or turbulent) boundary layers. For this course, we will hang on to the constant density assumption, as stated. For compressible flows, a compressible version of the Bernoulli's equation exists—which derivation requires the concept of entropy (thermodynamics!) be understood.

   - Do problem 4.6 on page 159 (diagram on page 161). You are expected to know that when the conduit in question has a sudden expansion, and the flow downstream at station #4 is nice and uniform again, quiet a bit of turbulent mixing must have occurred as the consequence of the sudden expansion.
   - Do problem 4.12 on page 161. If the flow is from left to right, you expect a jet coming out. If the flow is from right to left (you have essentially a vaccum cleaner), the sketch of the flow near station #2 would be different.
   - Do problem 4.25 on page 165. Assume the tank is big so that the flow is "quasi-steady," and the Reynolds number of the flow is very large. Use an intelligently chosen control volume!

4. Read Chapter 5 and remind yourself what you had learned in MAE 223. In addition, I have included additional supplementary materials below.

Supplementary comments to help your reading, and problem assignments.

- What is pressure? It is usually denoted by p. First of all, pressure is a scalar! It is not a vector because it has a magnitude but does not have a direction.

  A fluid mechanical definition of pressure is: the normal component of the force (vector) experienced by a very small (infinitesimal) surface element $d\mathcal{A}$ in contact with a fluid at rest is called the fluid pressure. What happens when the fluid is moving? the answer in the general case is somewhat complicated. For the moment, let us be satisfied by the assurance that the above description of pressure is a very good approximation when the fluid is moving—provided the Reynolds number of the fluid flow problem is not small. In addition, let us be satisfied by the assurance that our definition here is consistent with the discipline of thermodynamics.

- By exploiting the above definition of pressure, we can deal with the whole study of hydrostatics in a very elegant way, including the Archemedes Principle. Watch for the presentations in class.

- All physical laws are originally stated in terms of a *system*. A system is a collection of things you identified. For example, the conservation of mass says the mass of a system do not change. Newton says the gravitation force (vector) acting on his apple (his system of interest) equals to the time rate of change of the momentum (mass times velocity, a vector) of his apple. The First Law of Thermodynamics says the internal energy of a system is a state variable; if heat is added to the system and other kinds of energy (shaftwork or mechanical work, electricity, etc.) are added to the system, then the change of the internal energy of the system equals to the sum of the total energy added to the system.

- For fluid mechanics, we normally choose an arbitrary glob of fluid as our system.

- For fluid mechanics (as well as for many other fields of application), it is very useful to introduce the concept of a control volumne. What is a control volume? A control volume is a volume, fixed in space (*it does not move*) that **you** have chosen to be your very own. A smart engineer will choose a smart control volume? What is a smart control volume? A smart control volume is one which you have the best intuition (and concrete information) on what is going on at the surface of the control volume.

- We shall often need to write down the mathematical expression for the rate of change of some property Φ of a system. Consider $\Phi(x, y, z, t)$ to be the property of a very tiny glob of fluid which is moving with velocity $\mathbf{q} = (u, v, w)$?.[1] What is the rate of change of $\Phi(x, y, z, t)$ with respect to time? Since the glob moves, we know x, y, z are functions of time when we follow the glob around.

  Remember the **Chain Rule** of calculus? Using it, we obtain:

$$\begin{aligned} \left(\frac{d\Phi}{dt}\right)_{\text{system}} &= \frac{\partial \Phi}{\partial t} + \left(\frac{dx}{dt}\right)_{\text{system}} \frac{\partial \Phi}{\partial x} \\ &\quad + \left(\frac{dy}{dt}\right)_{\text{system}} \frac{\partial \Phi}{\partial y} \\ &\quad + \left(\frac{dz}{dt}\right)_{\text{system}} \frac{\partial \Phi}{\partial z} \end{aligned} \tag{1}$$

  All the fluid mechanicists in the world agreed to use the following short hand notation to represent the time rate of change of a system:

$$\frac{D}{Dt} \equiv \left(\frac{d}{dt}\right)_{\text{system}} \tag{2}$$

  This "operator", D/Dt, is usually called the *substantial derivative* (sometimes also known as the "material derivative," or the "Lagrangian derivative."). You must know its physical meaning: $D\Phi/Dt$ is the time rate of change of Φ of a tiny glob of fluid of your choice. Hence:

$$u = \frac{Dx}{Dt} \tag{3}$$

$$v = \frac{Dy}{Dt} \tag{4}$$

$$w = \frac{Dz}{Dt} \tag{5}$$

[1] from now on, u, v, w are the cartesian velocity components of the velocity vector $\mathbf{q}$— unless you are otherwise notified. Once in a while, we will use u to represent internal energy per unit mass of the fluid.

Using the above, we have:

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x} + v\frac{\partial \Phi}{\partial y} + w\frac{\partial \Phi}{\partial z}. \tag{6}$$

You may recall that, in cartesian coordinates, the *gradient* of Φ (a vector) is given by:

$$\nabla\Phi = \mathbf{e}_x\frac{\partial \Phi}{\partial x} + \mathbf{e}_y\frac{\partial \Phi}{\partial y} + \mathbf{e}_z\frac{\partial \Phi}{\partial z} \tag{7}$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are *unit vectors* in the x, y, z directions, respectively. The much more elegant vector form of the substantial derivative is:

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + \mathbf{q}\cdot\nabla\Phi \tag{8}$$

where $\cdot$ represents the dot in the dot product, an "operation" performed on two vectors.[2]

- It is most important that when you see a substantial derivative in an equation, you know what this term is telling you.
- An reminder: A property of a substance is called an intensive property if it has the same value for any subdivision of the system. For example, density (ρ, mass/volume), kinetic energy per unit mass ($(\mathbf{q}\cdot\mathbf{q})/2$, velocity-squared) are intensive variables.
- Now that we know how to find the time derivative of Φ associated with a tiny system (usually called an "infinitesical" system), how about when Φ is associated with a **BIG** (*i.e.* finite) system? For example, suppose I arbitrarily pick a BIG glob of fluid as my system at one moment in time. Let ϕ be an intensive variable with the unit of XXX/volume. Let Φ represent the total amount of XXX associated with the big, finite glob of fluid I had picked as my system. Hence:

$$\Phi(t) = \iiint_{\text{system}} \phi(x, y, z, t)d\mathcal{V} \tag{9}$$

[2]The dot product $\mathbf{v}\cdot\mathbf{q}$ is a scalar defined as $|\mathbf{v}||\mathbf{q}|\cos\theta$, where $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is physically meaningful and θ is the angle between the two vectors $\mathbf{v}$ and $\mathbf{q}$. In rectangular coordinates (x_1, x_2, x_3), we have $\mathbf{v}\cdot\mathbf{q} = v_1q_1 + v_2q_2 + v_3q_3$

where $d\mathcal{V}$ is $dxdydz$ in Cartesian coordinates, and represents an "infinitesimal" volume element. Note that after the volume integration has been carried out, $\Phi(t)$ is a function of t only. The QUESTION is: how do we go about computing the substantial derivative of $\Phi(t)$?

- The answer is called the *Reynolds Transport Theorem*:

$$\frac{D\Phi}{Dt} = \iiint_{\text{system}} \frac{\partial \phi}{\partial t} d\mathcal{V} + \iint_{\text{system}} (\mathbf{n} \cdot \mathbf{q}\phi) d\mathcal{A} \tag{10}$$

  where $\mathbf{n}$ is the *unit outward normal* of the surface element $d\mathcal{A}$ of the surface of the control volume—which is volume occupied by the original system at the moment we picked our system. I repeat: the control volume is the volume occupied by the system at the moment you picked the system. The control volume remains fixed in space while the system itself may be moving. Watch for its detailed derivation in class.

- The first term on the right hand side of (10) is called the *storage* term. It involves a volume integral over the control volume. It is zero for *steady* problems. The second term on the right hand side of (10) is called the *flux* term, or the convection term. It involves a surface integral over the surface of the control volume.

- The concept of flux is important and must be understood. In English, XXX flux is the amount of XXX moved across some surface of interest (fixed in space) per unit time. How does anybody move across this surface of interest? Because it has a velocity component perpendicular to this surface! Now consider a surface element $d\mathcal{A}$on this surface and denote its unit normal by $\mathbf{n}$. How much ϕ per unit time moved across this surface element in the $\mathbf{n}$ direction? The answer is: $(\mathbf{n}\cdot\mathbf{q}\phi)d\mathcal{A}$. The Φ for $\phi = 1$ is called volume flux, for $\phi = \rho$ is called the mass flux, for $\phi = (\mathbf{q} \cdot \mathbf{q})/2$ is called the kinetic energy flux, and for $\phi = \rho\mathbf{q}$ is called the momentum flux (a vector!).

- We shall have many occasions to work on surface integrals of various sort on the surface of control volumes. Here comes a little bit of fancy (but very important) math. It turns out that certain surface integrals can be converted into volume integrals. The following are two such cases:

$$\iint_{\text{system}} (\mathbf{n} \cdot \mathbf{v}) d\mathcal{A} = \iiint_{\text{system}} (\nabla \cdot \mathbf{v}) d\mathcal{V} \tag{11}$$

$$\iint_{\text{system}} (\mathbf{n}\varphi) d\mathcal{A} \ = \iiint_{\text{system}} (\nabla\varphi) d\mathcal{V} \tag{12}$$

where $\mathbf{v}$ is any continuous and differentiable vector field, and φ is any continuous and differentiable scalar field. Both are usually called *divergence theorems*; sometimes only the first one is called divergence theorem, while the second one is called divergence theorem for gradients. Actually, called either a theorem is really a misnomer: both are actually *definitions*: (11) is the definition of the divergence operator $\nabla\cdot$ which operates on vectors, and (12) is the definition of the gradient operator ∇ which operates on scalars. The actual operational details of these operators depends on the coordinate system used. In Cartesian coordinates, we have:

$$\nabla \cdot \mathbf{v} \ = \ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}, \quad (\nabla \cdot \mathbf{v} \text{ is a scalar}) \tag{13}$$

$$\nabla\varphi \ = \ \mathbf{e}_x \frac{\partial\varphi}{\partial x} + \mathbf{e}_y \frac{\partial\varphi}{\partial y} + \mathbf{e}_z \frac{\partial\varphi}{\partial z}, \quad (\nabla\varphi \text{ is a vector}), \tag{14}$$

where $\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + \mathbf{e}_z v_z$. If you need $\nabla \cdot \mathbf{v}$ or $\nabla\varphi$ in other coordinates (*e.g.* cylindrical polar, spherical, ..., etc.), you either work them out using the definitions, or look them up in text books.

- In Chapter 2, you will have occasions to find the moment $\mathbf{M}$ (vector) of a force vector $\mathbf{f}$ acting at a point with position vector $\mathbf{r}$ drawn from the origin. In English, moment equals to the moment arm times the force; the "direction" of $\mathbf{M}$ is determined (by unanimous consent) by the right hand rule (use your right hand's four fingers to point in the direction of $\mathbf{f}$ first, then bend them to point in the direction of $\mathbf{r}$; your thumb is now pointing in the direction of $\mathbf{M}$). The moment arm is the "perpendicular distance" from the origin to the $\mathbf{f}$ vector. Mathematically, we have $\mathbf{M} = |\mathbf{r}||\mathbf{f}| \sin\theta$ where θ is the angle between the two vectors $\mathbf{r}$ and $\mathbf{f}$. The fancy notation is $\mathbf{M} = \mathbf{r} \times \mathbf{f}$, called the *cross product* of $\mathbf{r}$ with $\mathbf{f}$. When $\mathbf{r}$ and $\mathbf{f}$ are in Cartesian coordinates (x_1, x_2, x_3), we have:

$$\mathbf{r} \times \mathbf{f} = \text{determinant of} \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ r_1 & r_2 & r_3 \\ f_1 & f_2 & f_3 \end{vmatrix} \tag{15}$$

- Some of you may have encounter the concept of the curl of a vector $\mathbf{v}$ in your math courses, usually denoted by $\nabla \times \mathbf{v}$. We temporarily do not need to worry about this yet. Its time will come.