MAE 224 Notes #4a
Elements of Thermodynamics and Fluid Mechanics

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1 Reading and Homework Assignments

The problems are due on Wednesday, March 3rd, 1999, 5PM. Please submit your homework to the MAE 224 homework **IN** tray outside D-302, E.Q.

- Read Chapter 11, Open Channel Flow; pp. 375-403 in Professor Smits' *A Physical Introduction to Fluid Mechanics.*

2 More to Come

Wait for Notes 4b for comments on the readings and problem assignments. I hope to post them by wednesday, Feb. 24th.

3 Major Control Volume Concepts

The big deal is: **control volume**. We learned how to express the Laws of Physics, which are originally stated in terms of systems, into the language of control volumes. Remember, a system is a collection of matter that you identified as your system. A control volume is a volume fixed in space that you have selected to be your control volume.

Conservation of Mass In English, it says the mass of a system does not change with time. In terms of systems, the mathematical way of saying it is:

$$\frac{D}{Dt}\iiint_{\text{system}} \rho d\mathcal{V} = 0. \tag{1}$$

Here, D/Dt is the notation for substantial derivative—the rate of change with respect to time of an identified system. In terms of control volumes, the mathematical way of saying this Law of physics is (here $\mathbf{q}$ is fluid velocity vector):

$$\iiint_{\text{CV}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \iint_{\text{surf.CV}} \rho(\mathbf{n}\cdot\mathbf{q})d\mathcal{A} = 0. \tag{2}$$

The first term on the left hand side is called the storage term, and is zero for steady problems. The second term on the left hand side is called the flux or convective term; $\mathbf{n}$ is the unit **outward** normal of the surface element $d\mathcal{A}$, and $\mathbf{q}$ is the fluid velocity vector as mentioned previously.

In steady flow, the conservation of mass in terms of control volume is:

- the net mass flux out of a control volume is zero.

Or,

- whatever mass flux comes into a control volume, it must go out.

Or, for steady flow in a streamtube:

- the mass flow rate is a constant along the streamtube.

If we use the divergence theorems (see Notes 3a), this equation becomes:

$$\iiint_{\text{CV}} \left\{ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{q}) \right\} d\mathcal{V} = 0. \tag{3}$$

Since (3) is valid for arbitrary choice of the control volume, the curly bracketed integrand must be zero everywhere. That is our Law of Conservation of Mass in differential equation form. The $\nabla\cdot$ is the notation for the divergence operator. You should know by heart what it is in Cartesian coordinates. Look up your math book on what it is in different coordinates.

Conservation of Momentum In English, it says: the resultant external force acting on a system equals the rate of change of momentum of the system. In terms of system, the mathematical way of saying it is:

$$\iint_{\text{surf.CV}} -\mathbf{n} p d\mathcal{A} + \iiint_{\text{system}} \rho \mathbf{g} d\mathcal{V} = \frac{D}{Dt} \iiint_{\text{system}} \rho \mathbf{q} d\mathcal{V}. \qquad (4)$$

The left hand side is the sum of all the external forces. The external force due to viscous effects has been neglected (because the Reynolds number of the problem is expected to be large). The first term is the contribution due to pressure. The second term is the gravitational force exerted on the system by mother earth. Note that the vector $\mathbf{g}$ can be represented (on a flat earth) by $-g\nabla z$ where z is the altitude of the point of interest. If there are other external forces exerted by physical objects which "cross" the boundary of the system, they must be included. In terms of control volume, the mathematical way of saying this Law of physics is (forces due to pressure and gravity only):

$$\iint_{\text{surf.CV}} -\mathbf{n} p d\mathcal{A} \quad + \quad \iiint_{\text{CV}} \rho \mathbf{g} d\mathcal{V} \qquad (5)$$

$$= \iiint_{\text{CV}} \frac{\partial(\rho \mathbf{q})}{\partial t} d\mathcal{V} \quad + \iint_{\text{surf.CV}} \rho \mathbf{q} (\mathbf{n} \cdot \mathbf{q}) d\mathcal{A} \qquad (6)$$

This is a vector equation, the English version of $\mathbf{f} = m\mathbf{a}$ using the language of control volume concepts. The first term on the right hand side is zero for steady flow problems. For steady flow problems: the net external resultant force (vector!) acting on a control volume equals to the net outflux of momentum (vector!).

If we use the divergence theorems (both of them are needed; see Notes 3a), this equation becomes (with the help of the continuity equation just derived):

$$\iiint_{\text{CV}} \left\{ -\nabla p + \rho \mathbf{g} = \rho \frac{D\mathbf{q}}{Dt} \right\} d\mathcal{V}. \qquad (7)$$

Each of the terms in the curly bracket is a vector.[1] The ∇ here is the notation for the gradient operator. ∇p is a vector—here it has the physical meaning of "net pressure force" per unit volume. You should know by heart what it is in Cartesian coordinates. Look up your math book on what the gradient operator is in different coordinates.

[1] Remember, if you have a vector equation, each of the terms in the equation must be a vector.

Conservation of Energy You should be able to do this yourself—using symbols to represent contributions of heat or shaftwork added to the (arbitrarily chosen by you) system of interest. Of course, if the boundary of your system moves, the work done per unit time by pressure on the boundary of your system (called the pressure work) must be included (with the proper sign!). For moving systems, the internal energy per unit mass consists of its thermal energy (denoted by u in Smits), and its kinetic energy per unit mass ($q^2/2$). To get the total internal energy of your chosen system, you need to integrate.

Reynolds Transport Theorem You will often need to evaluate the time derivative of something associated with a system (Φ is something per unit volume):

$$\frac{D}{Dt}\iiint_{\text{system}} \Phi d\mathcal{V}. \tag{8}$$

Osborne Reynolds (the same guy that is honored by the Reynolds number) translated this using the language of control volume many years ago, and his result is called the Reynolds Transport Theorem. We have already used it twice in the above expositions:

$$\frac{D}{Dt}\iiint_{\text{system}} \Phi d\mathcal{V} = \iiint_{\text{CV}} \frac{\partial \Phi}{\partial t} d\mathcal{V} + \iint_{\text{surf.CV}} \Phi(\mathbf{n}\cdot\mathbf{q}) d\mathcal{A}. \tag{9}$$

The first term on the right hand side is zero when the problem is steady. The second term on the right hand side is the "net outflux of Φ" from the surface of the control volume. "Outflux" means flux going out is postive.

If you need to find the substantive derivative of a vector property of a system (such as the momentum of a big glob of fluid), the best thing to do is to represent your vector in Cartesian coordinates, and then apply the Reynolds Transport Theorem on each cartesian component of the vector. **Warning**: this procedure gives you the **WRONG** answers if you use it on a vector represented in non-Cartesian coordinates (*i.e.* cylindrical polar). If you need to use non-Cartesian coordinates, look up the vector form of the Reynolds Transport Theorem from any fluid mechanics text book. (I would welcome the question in class: sir, how would you derive the vector version?)

For those who remember what a curl is, the following vector identity is helpful when you want the substantial derivative of momentum:

$$\mathbf{q} \cdot \nabla \mathbf{q} = \frac{1}{2} \nabla (\mathbf{q} \cdot \mathbf{q}) - \mathbf{q} \times (\nabla \times \mathbf{q}). \tag{10}$$

We won't need this until a couple of weeks from now.

What Control Volume Should I pick? The control volume you pick tells the world how smart you are. The wise men/woman always pick a control volume whose surface he/she has the best intuition on what is going on—so that the surface integrals (and their unit outward normals) cause you little difficultis to evaluate.

The other big deal is Bernoulli's equation. It is:

$$\frac{p}{\rho} + \frac{\mathbf{q}^2}{2} + gz = \text{constant} \tag{11}$$

where z is the altitude of the point of interest. Before using it, make sure the following **FOUR** assumptions are satisfied:

- The flow is inviscid—the Reynolds number is large and viscous friction is negligible
- The flow is steady
- The two stations of interest are on the same streamline
- The flow is incompressible—the expected density change in the flow field is very small.

The last assumption is always in good shape when water is the fluid. For air, the last assumption is OK when the flight Mach number is small (such as when an airplane is taking off). When you know your thermodynamics and understand entropy, the last assumption can be modified to yield a high Mach number version of Bernoulli's equation—in MAE 335.

Bernoulli's constant is not a constant along a streamline that goes through an unsteady flow region, such as a propeller, a compressor, or a turbine. The reaon is simply that the flow is unsteady (put your finger in there and you will confirm that the blades are moving around!). The Bernoulli's "constant" changes across such regions.

Most importantly, the Bernoulli's equation does not hold when your "streamline" goes through a region of "turbulent mixing." In general, the Bernoulli's constant DROPS when your streamline goes through any experience of dissipation. Most importantly, the Bernoulli's equation is NOT an energy equation. For example, if you have an incompressible fluid and you heat it, Bernoulli's equation could not care less.

3.1 Fluid pressure at exit plane

If you have fluid coming out of a tube into a big environment of fluid at rest, experience tells us that we expect a **jet**. For example, if you run a vacuum cleaner in "reverse," you have a jet coming out of the end of your bare hose (no attachment). If you have reasons to believe that the jet's boundary is straight and parallel (more or less) and the fluid outside the jet is quiescent (more or less), then the pressure inside the jet near the exit plane is approximately the static pressure of the undisturbed environment. In other words, when you have fluid issuing into a quiescent environment, the fluid pressure in the jet near the exit plane is the same as the pressure in the undisturbed environment. The Bernoulli's constant on a streamline coming out of the hose is larger than the Bernoulli's constant evaluated on any quiescent undisturbed fluid element in the environment (when gravity is neglected). If you follow a fluid element after it comes out of the exit plane, you expect to see it eventually mixes with the environment's fluid via turbulent or laminar mixing, and gradually slows down *without* concurrent rise in pressure. This mixing process is a dissipative experience for the fluid elements, and the Bernoulli's constant is degraded as one moves downstream.

If you run a vacuum cleaner the "normal way" with a bare hose, air is sucked into the bare hose. The air does NOT enter the bare hose as a jet; rather, it comes from all around. The streamlines are curved. The pressure of the entering fluid at the exit plane of the bare hose is NOT the undisturbed pressure of the environment. It is LOWER. That is why the dirts in the vicinity of the end of the bare hose go toward the entrance of the bare hose! The Bernoulli's constant on any streamline entering the hose is the same as the Bernoulli's constant evaluated on any quiescent fluid element in the environment (when gravity is neglected).