MAE 224 Notes #4b
Elements of Thermodynamics and Fluid Mechanics

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1 Reading and Homework Assignments

The problems are due on Wednesday, March 3rd, 1999, 5PM. Please submit your homework to the MAE 224 homework **IN** tray outside D-302, E.Q.

- Read Chapter 11, Open Channel Flow, pp. 375-403, in Professor Smits' *A Physical Introduction to Fluid Mechanics.*

2 Comments on Readings and Problems

1. §11.1, Introduction. Open channel flow is the study of the free surface between air and a flowing liquid.

2. §11.2, Small amplitude waves. We know that when a pebble is tossed into a shallow pond, a ripple is observed to propagate away from the point of impact with velocity c. If we are told that c depends on the depth H of water in the pond, and the gravitational acceleration (of the earth) g, what could dimensional analysis do for you? You will find $\Pi_1 = c/\sqrt{gH}$. In the open channel world, Π_1 is denoted by F, and is called Froude Number, honoring Mr. Froude who studied this field of scholarship many years ago. Since c is the speed of propagation of a "disturbance," Fig. 11.3 becomes intuitively comfortable.

3. §11.3 Froude Number. Once F is defined, its physical meaning is obvious: it is the ratio of kinetic energy per unit mass to gravitational energy per unit mass.

4. Once you realize that c is a monotonically increasing function of H, the conclusion that a smooth wave will steepen and eventually "break" is a logical deduction.

5. 11.5, Tsunamis. Japenese for *tidal waves*. Fun readings.

6. §11.6, Hydraulic Jumps. Fig. 11.10 shows you the generic problem of a hydraulic jump. At station #1 you have the incoming flow with velocity U_1 and water height H_1. What are the possible conditions at station #2, when nothing external is happening inbetween? Well, one obvious solution is: condition at station #2 is identical to station #1. When we ask the Laws of Physics (conservation of mass and momentum), we find there is another possibility! The water runs into a mess of foams and turbulence, coming out the right hand side slower and with a larger depth. This is a hydraulic jump. Most important observation: the Bernoulli's constant is not a constant across a hydraulic jump! It always decreases. The derivation of (**??**)11.8) on page 387 is straightforward—the dimensionless water height ratio is found to be a function of the dimensionless Froude number *in front* of the hydraulic jump.

7. §11.7, Hydraulic Drops? No hydraulic drop is ever observed, and it is NOT ALLOWED. Why? if it actually occurs, you will find Bernoulli's constant spontaneously increases across the "drop." You can exploit this to make yourself into a millionaire! In all seriousness, hydraulic drop violates the Second Law of Thermodynamics.

8. §11.8 Surges and Bores. Hydraulic jump as analyzed in the previous section is observed as a steady flow. The position of the jump is fixed in space. When you observe a moving hydraulic jump, you oftern call them surges and bores.

9. §11.9, 11.9.1 and 11.9.2. Flow Through a Smooth Constriction. What happens when the sides of a rectangular channel has a smoothly narrowing and then widening section? The response of the free surface depends on the upstream Froude number! Look at (11.17) on page 393.

10. §11.9.3. Flow over bumps. What happens when a shallow stream flows over a smooth bump on the bottom of the channel? If the upstream Froude numer is subcritical (*i.e.* $F_1 < 1$), the free surface dips on the bump. If the upstream Froude numer is supercritical (*i.e.* $F_1 > 1$), the free surface rises and falls with the bump. I will show in class how to compute the volume flow rate over a smooth "weir" whose top is lower than the free surface of a lake.

11. 11.9.4, Summary and Examples. Study them. Look at example 11.2: draw a smart control volume, and apply the balance of momentum—the net external force acting on the control volume equals to the net outflux of momentum. Look at Fig. 11.20. The artist who drew this picture should show y_4 to be definitely smaller than y_1 (and you know why!). Example 11.4 is fun. You can solve the problem of a gate suddenly appearing and blocking a flowing stream!

3 Supplementary Notes

We consider a horizontal rectangular channel with water flowing from left to right. Let x be the "streamwise" coordinate. The width of the channel is $W(x) > 0$, and $W = W_o$ at $x = 0$. The geometry of the bottom elevation is given by $z = h(x)$. The elevation of the bottom of the channel at $x = 0$ is used as a reference level so that $h(x = 0) = 0$. For $x \neq 0$, both $W(x)$ and $h(x)$ are smooth. Note that $h(x)$ may be positive (there is a maximum allowable value), and may be negative (when the channel water depth is very deep).

We have steady flow.

Let $u(x)$ denotes the flow velocity at position x. Let $y(x)$ denotes the depth of the water at position x—the distance between the local free surface and the local bottom of the channel. The x-velocity of the water is u_o and the water depth is y_o at $x = 0$. (remember, the actual location chosen where $x = 0$ is at YOUR disposal).

The continuity equation is (cancelling ρ because it is a constant):

$$u_o W_o y_o = u(x) W(x) y(x) \tag{1}$$

The elevation of the free surface is $h(x) + y(x)$. The Bernoulli's equation when applied to the surface streamline yields (the gage pressure at the free surface is zero):

$$g\left(h(x)+y(x)\right)+\frac{1}{2}u^2(x)=gy_o+\frac{1}{2}u_o^2. \tag{2}$$

Now, define the local Froude number F by:

$$F=F(x)=\frac{u(x)}{\sqrt{gy(x)}}. \tag{3}$$

Lets get rid of $u(x)$ in favor of $F(x)$. We have:

$$\sqrt{g}F_oW_oy_o^{3/2} = \sqrt{g}F(x)W(x)y^{3/2}(x), \tag{4}$$

$$gy_o(1+\frac{1}{2}F_o^2) = gh(x)+gy(x)\left(1+\frac{1}{2}F^2(x)\right). \tag{5}$$

Solving for $y(x)$ from (5) above, we have:

$$y(x)=\frac{y_o(1+\frac{1}{2}F_o^2)-h(x)}{1+\frac{1}{2}F^2(x)} \tag{6}$$

Using this in (4) above, we have (after some rearrangements):

$$F_oW_o=W(x)\left[(1+\frac{1}{2}F_o^2)-\frac{h(x)}{y_o}\right]^{3/2}D(F(x)) \tag{7}$$

where $D(F)$ is given by:

$$D(F)\equiv\frac{F}{\left(1+\frac{1}{2}F^2\right)^{3/2}}. \tag{8}$$

The shape of the D vs. F curve is very simple: it starts at $D=0$ for $F=0$, rises to a maximum at $F=1$, and falls for $F>1$, approaching zero as $F\to\infty$.

With equation (8) in hand, and armed with a good graph of D versus F, you can determine how the Froude number respond to smooth side constrictions (as described by $W(x)$) and smooth bottom bumps (as described by the square bracket). A succinct way to summarize the the situation is: if the value of D increases, then the Froude number is moving toward unity; if the value of D decreases, then the Froude number is moving away from unity.

Problems on pp. 403-416.

1. Problem 11.9. This channel has constant width. This problem is a special case of what I presented above, but the analysis proceeds using differential equations instead of algebraic equation. Starting with the same continuity equation and Bernoulli's equation, you differentiate both of these equations with respect to x, and try to get rid of u in favor of F.

2. Consider the Figure of Problem 11.13, but do NOT do the problem given there. Do the following problem instead. Let the far upstream depth to be very, very large. Let the elevation of the upstream free surface be Z meters above the peak of the obstacle. (referring to my presentations on the previous pages, where would you pick as your $x = 0$ station?)

    - The free surface is observed as shown. Provide arguments to justify the conclusion that the Froude number at the peak of the obstacle is unity (the Froude number at the far left is very, very small),
    - Find y_2 as a function of Z.
    - Explain in English what is happening on the right hand side if there is a tall vertical wall there.

3. Problem 11.27. You need to change your coordinates so that the bore is stationary as observed in the new moving coordinates. Now you just have a simple steady hydraulic jump.

4. Problem 11.33. Use the continuity equation, always assume the velocity at any station is uniform. Draw a control volume, and apply the conservation of mass: whatever mass flows into the control volume must flow out. Once you find the Froude number at station #1, the hydraulic jump formula can be used under the assumption that the "thickness" of the jump is small.