MAE 224 Notes #5a
Elements of Thermodynamics and Fluid Mechanics

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1 Reading and Homework Assignments

The problems are due on Wednesday, March 10rd, 1999, 5PM. Please submit your homework to the MAE 224 homework **IN** tray outside D-302, E.Q.

- Read Chapter 7, Incompressible Irrotational Flows; pp. 247-275 in Professor Smits' *A Physical Introduction to Fluid Mechanics.*

1.1 Notice of Midterm

Mid Term next Friday in class.

2 Comments on Vorticity

An alternative title for this chapter is: *Low Speed Aerodynamics.* We will learn why airplanes are such wonderful machines.

The most striking thing about airplanes is that the weight it can carry is much larger than the thrust you must provide in flight (large lift-to-drag ratio). You have noticed that subsonic wings have round and smooth leading edges and always a sharp trailing edge. You have noticed that long range aircrafts have long wing spans in comparison to their wing chords (large aspect ratio). What we are about to learn this week provide some answers to questions that you may have on these observations.

The big deal here is *vorticity*, a vector denoted by $\mathbf{\Omega}$. Its definition is:

$$\mathbf{\Omega} \equiv \nabla \times \mathbf{q} \tag{1}$$

where $\mathbf{q}$ is the fluid velocity vector. The $\nabla\times$ is called the *curl* operator—it is an instruction for you to make the $\mathbf{\Omega}$ from a given $\mathbf{q}$. In Cartesian coordinates, the explicit instruction is:

$$\nabla \times \mathbf{q} = \text{determinant of} \quad \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{2}$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are unit vectors in the x, y, z directions, respectively, and u, v, w are the x, y, z components of $\mathbf{q}$, respectively. If someone gave you $\mathbf{q}(x, y, z; t)$, you can compute the vorticity field immediately from the given information.

But why should be we interested in vorticity?

It turns out that for typical problems in low speed aerodynamics (*e.g.* an airplane in steady low speed flight in an otherwise undisturbed atmosphere), we can prove that the vorticity field is zero nearly everywhere. In other words, the typical, generic low speed aerodynamic flow field is nearly vorticity free! Such flows are called irrotational flows, because vorticity has the physical meaning of being twice the average angular velocity vector of an element of fluid. See §7.1.

Watch for the proof (and conditions of) for a flow field to be irrotational in class. The conditions are:

1. Viscous effects can be neglected; *i.e.* the Reynolds number of the problem is large,
2. The density ρ is a constant,
3. Once upon a time, the fluid in the domain of interest had no vorticity.

Hence, typical low speed aerodynamics give rise to irrotational problems because of large Reynolds number, constant density (consequence of low speed), and that the fluids elements far, far upstream were clearly irrotational.

Think about the following problem. If you are given a FRICTIONLESS, perfectly smooth rigid sphere made of CONSTANT DENSITY material which is INITIALLY NOT SPINNING, can you make it spin? Compare the capitalized words with the irrotational conditions.

Instead of working out vorticity after you have the velocity field $\mathbf{q}(x, y, z; t)$, the shoe is now on the other foot. Now that we know the flow has no vorticity, how does this information help us find the velocity field?

The answer is: it helps tremendously!

In your math courses, you may have learned, or have already learned, that the curl of a gradient is always zero. The relation

$$\nabla \times (\nabla \phi) = 0 \tag{3}$$

is a mathematical identity. It is always true. There is another mathematical identity:

$$\nabla \cdot \mathbf{\Omega} = 0. \tag{4}$$

This innocent looking equation is equivalent to the English sentence: a vortex line cannot end in a fluid. (a vortex line is a line whose tangent at every point is parallel to the local vorticity vector).

So here comes the first dividend. Instead of looking for a vector field $\mathbf{q}(x, y, z; t)$, which involves looking for its three components, we can introduce the single unknown scalar $\phi(x, y, z; t)$ by:

$$\mathbf{q} = \nabla \phi. \tag{5}$$

In Cartesian coordiantes, this is:

$$u = \frac{\partial \phi}{\partial x} \tag{6}$$

$$v = \frac{\partial \phi}{\partial y} \tag{7}$$

$$w = \frac{\partial \phi}{\partial z} \tag{8}$$

This way, the $\mathbf{q}$ so generated is automatically irrotational. We have succeeded in replacing a vector unknown (with three components u, v, w) into a single scalar unknown, ϕ.

But, how do we go away finding ϕ for my own aerodynamics problem?

The incompressible continuity is:

$$\nabla \cdot \mathbf{q} = 0. \tag{9}$$

In Cartesian coordinates, this is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{10}$$

Using $\nabla\phi$ to replace $\mathbf{q}$, the continuity equation can be rewritten as:

$$\nabla \cdot (\nabla\phi) = 0, \tag{11}$$

which is called the *Laplace Equation.* In Cartesian coordinates, the Laplace equation is:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{12}$$

So, the task of solving an aerodynamics problem now reduces to solving the Laplace equation. See §7.7.

How do we go about looking for solutions of Laplace equation? We take advantage of the observation that the Laplace equation is *linear*, and allows superposition of solutions. In other words, if ϕ_1 and ϕ_2 are known solutions of the Laplace equation, then $C_1\phi_1 + C_2\phi_2$ is also a solution of the Laplace equation (where C_1 and C_2 are constants). So here is the trick. All fluid mechanics books give you a short list of *elementary solutions.* You can just add them together to create irrotational flows fields that you like. The short list usually includes: uniform flow, source or sink, doublet, and the line vortex (watch for this in class). We will show in class how to play with these elementary solutions.

What do you do after you have found the irrotational flow field that you are interested in, and now you want to find the associated pressure field? For steady flow problems, the Bernoulli's equation is at your service (usually, the gravity term gz is neglected because most airplanes are small in comparison to $28,000$ feet). Once you know your $\phi(x, y, z; t)$, you can find $\mathbf{q}(x, y, z; t)$, and through the Bernoulli's equation you can find the static fluid pressure. It is helpful to know that, for irrotational problems, the Bernoulli's constant is the same on every streamline.

What if after you have found the velocity field and the associated pressure field, you are now interested in the lift and drag of your balloon or your airplane? You can use the big control volume and do a momentum balance.

Here comes the punch lines: if we hang on to the conclusion that there is no vorticity anywhere in the flow field, the irrotional theory says there is no net force acting on any finite body. This is called the D'Alembert Paradox. If there is a line vortex "inside" your wing, then there is a lift force per unit length of the vortex, but there will be a vortex wake. How did a line vortex get inside your wing? What does the vortex wake do to the D'Alembert Paradox? We will have a fun Q and A session in class.

In the mean time, you need to do some more readings and do some problems.

3 Comments on Reading and Problems

1. Read §7.1 and 7.2. Skip §7.3 and §3.4. The concept of stream function ψ is only useful for two-dimensional and axi-symmetric flows. So we will skip it because of time limitations.
2. §7.6. Here comes the short list of elementary solutions.
3. §7.7. The Laplace Equation derived.
4. §7.8, §7.9. Application of the principle of superposition for linear problems. I take a bit of this, a bit of that, and see what I get from the sum.
5. §7.10. Lift. Yes, there can be a lift, but only if there is a line vortex. Equation (7.32) is the big deal.
6. §7.11. Trailing vortex system! This is the reason why long range aircrafts have long wings.

Problems:

1. Problem 7.19 on page 278. For part (a), skip the stream function. Just do the velocity potential.
2. Problem 7.21 on page 278. Put the source at the origin of your coordinate system. Replace part (d) by:

    > (d) Find the net external force needed to hold the source in place. Hint: use a big, big, big control volume (which includes the source) so that the pressure contributions cancel out. So the net external force acting on this control volume equals to the net momentum outflux. Is there a net mass outflux? Yes. How much net x-momentum is being outflux along with the net mass outflux? Look at your class notes to see how I managed to reduce the surface calculations to just the rear surface.