MAE 224 Notes #5b
Elements of Thermodynamics and Fluid Mechanics

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The fact that high Reynolds number, low speed aerodynamic flows (uniform flows over "obstacles") are irrotational (zero vorticity almost everywhere) is a major theoretical simplification. I presented the intuitive physical basis of this result in class. In addition, I gave in class a very brief and somewhat water-down version of the mathematical derivation of this result. Because of the importance of this result, I will present below a more precise verion of the mathematical theory. **You are not responsible for this material in the mid term exam. If you got lost by the math, don't worry about it**. The important thing to remember is the physical reasonings which leads to the final results, and the conditions under which the conclusion drawn is valid. What is presented below is just an elegant way of saying the physical (and intuitive) ideas using the formal language of mathematics.

1 Substantial Derivative and Other useful Vector Identities

The substantial derivative D/Dt is the rate of change with respect to time following an element of fluid (our tiny system). In vector form, it is given by:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla. \tag{1}$$

where $\mathbf{q}$ is the fluid velocity vector. In Cartesian coordinates, it is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial x} + w\frac{\partial}{\partial x}, \tag{2}$$

where u, v, w are the x, y, z components of $\mathbf{q}$, respectively.

Let $\mathbf{A}$ and $\mathbf{B}$ be two vector fields. The following are useful identities:

$$\mathbf{A} \cdot \nabla\mathbf{A} = \nabla\left(\frac{\mathbf{A}\cdot\mathbf{A}}{2}\right) - \mathbf{A} \times (\nabla \times \mathbf{A}), \tag{3}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{B} \cdot \nabla\mathbf{A} - \mathbf{A} \cdot \nabla\mathbf{B}. \tag{4}$$

Using $\mathbf{A} = \mathbf{q}$ and $\mathbf{B} = \mathbf{\Omega} = \nabla \times \mathbf{q}$, we obtain:

$$\mathbf{q} \cdot \nabla\mathbf{q} = \nabla\left(\frac{\mathbf{q}\cdot\mathbf{q}}{2}\right) - \mathbf{q} \times \mathbf{\Omega}, \tag{5}$$

$$\nabla \times (\mathbf{q} \times \mathbf{\Omega}) = -\mathbf{\Omega}(\nabla \cdot \mathbf{q}) + \mathbf{\Omega} \cdot \nabla\mathbf{q} - \mathbf{q} \cdot \nabla\mathbf{\Omega}. \tag{6}$$

In the last equation, we had taken advantage of $\nabla \cdot \mathbf{\Omega} = 0$.

2 The Momentum Equation (Horace Lamb)

The momentum (vector) equation of fluid mechanics is:

$$\frac{D\mathbf{q}}{Dt} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{q} - g\nabla z \tag{7}$$

where ν is the kinematic viscosity of the fluid and z is the altitude of the point in question, and g is the magnitude of the gravitational acceleration of the (flat) earth. The middle term on the right hand side of (7) represents viscous effects. For the sake of simplicity, we have assume that ν is a constant.

The left hand side of (7) can be rewritten using the vector identifies:

$$\frac{D\mathbf{q}}{Dt} = \frac{\partial\mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla\mathbf{q} \tag{8}$$

$$= \frac{\partial\mathbf{q}}{\partial t} + \nabla\left(\frac{\mathbf{q}\cdot\mathbf{q}}{2}\right) - \mathbf{q} \times \mathbf{\Omega}. \tag{9}$$

Equation (7) can be rewritten as:

$$\frac{\partial\mathbf{q}}{\partial t} + \nabla\left(\frac{\mathbf{q}\cdot\mathbf{q}}{2}\right) - \mathbf{q} \times \mathbf{\Omega} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{q} - g\nabla z. \tag{10}$$

This is called the Lamb's form of the momentum equation.

When the viscous term is neglected, when density is a constant and when the flow is steady, (10) simplifies to:

$$\nabla \left(\frac{p}{\rho} + \frac{\mathbf{q} \cdot \mathbf{q}}{2} + gz \right) = \mathbf{q} \times \Omega. \tag{11}$$

For irrotational flow fields, we have $\Omega = 0$. Hence, we have:

$$\nabla \left(\frac{p}{\rho} + \frac{\mathbf{q} \cdot \mathbf{q}}{2} + gz \right) = 0 \tag{12}$$

which, upon integration, yields:

$$\frac{p}{\rho} + \frac{\mathbf{q} \cdot \mathbf{q}}{2} + gz = B \tag{13}$$

where B is a constant (independent of x, y, z), and has the physical meaning of a Bernoulli's constant.

Hence, we have just proven that the Bernoulli's constant for an inviscid, constant density, irrotational steady flow is the same constant for all streamlines in the flow.

3 The Vorticity Equation

We now take the curl of Lamb's form of the momentum equation, taking advantage of the fact that the curl of a gradient is always zero. We obtain:

$$\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{q} \times \Omega) = -\nabla \left(\frac{1}{\rho} \right) \times \nabla p + \nu \nabla^2 \Omega \tag{14}$$

Using (6), we have:

$$\frac{D\Omega}{Dt} - \Omega \cdot \nabla \mathbf{q} + \Omega (\nabla \cdot \mathbf{q}) = -\nabla \left(\frac{1}{\rho} \right) \times \nabla p + \nu \nabla^2 \Omega \tag{15}$$

The $(\nabla \cdot \mathbf{q})$ factor on the left hand side can be eliminated by using the continuity equation:

$$\frac{D\rho}{Dt} + \rho (\nabla \cdot \mathbf{q}) = 0. \tag{16}$$

Finally, we have (after a bit of manipulations):

$$\frac{D}{Dt}\left(\frac{\Omega}{\rho}\right) = \frac{\Omega}{\rho} \cdot \nabla \mathbf{q} + \frac{\nabla \rho \times \nabla p}{\rho^3} + \frac{\nu}{\rho}\nabla^2 \Omega. \tag{17}$$

This equation is called the *vorticity equation*. It tells us that the Ω/ρ vector following an element of fluid (our tiny system) can change by three different mechanisms: (1) the first term—when the current Ω/ρ vector is being distorted by nonuniform local velocity field, (2) the second term—by the so-called baroclinic term, and (3) the third term—by viscous effects. If density is a constant, the baroclinic term is zero (it is very important for geophysical fluid dynamics—the mechanics of the atmosphere and oceans).[1] If the flow have large Reynolds number, the viscous term is negligible. Finally, if all the fluids elements in our flow field of interest were known to have zero Ω once upon a time, and since then no baroclinic and no viscous effects had been experienced , then the region occupied by such fluids elements is irrotational.

As I said, do not worry about it if the math used above is too messy for you. What I want you to understand is that the conclusion "my flow is irrotational" is a "derived conclusion." A generic low speed aerodynamics problem usually satifies the requirements of the irrotational conclusion.

[1] Actually, if somehow one can show that one thermodynamic variable of the whole flow field is a constant (such as entropy), the baroclinic term is also zero then.