Problem 19, Page 128

The characteristic equation for $\frac{d^2y}{dt^2} - y = 0$ is $r^2 - 1 = 0$. This has two distinct roots $r = 1$ and $r = -1$, thus the solution is a linear combination of $e^t$ and $e^{-t}$. That is to say $y(t) = c_1e^t + c_2e^{-t}$. We find $c_1$ and $c_2$ by using the boundary conditions. We have $y(0) = c_1 + c_2 = 5/4$ and $\frac{dy}{dt}(0) = c_1 + c_2 = -3/4$, giving $y(t) = \frac{1}{4}e^t + e^{-t}$. To find the min, we set $\frac{dy}{dt} = 0$, solving, we find $t = \ln(2)$. The verification that this is a max can be done by looking at the second derivative or the properties of the first derivative. The hyperbolic trig functions $\sinh(t)$ and $\cosh(t)$ are both solutions, just do the derivatives. If you didn’t use a computer to the plot this time, we let you off. But that would happen again! Learn to do it; it’s good for you in the long run.

![Figure 1: Solution for Problem 19, Page 128](image-url)
Problem 30, Page 128

To solve \( \frac{d^2y}{dt^2} + t(\frac{dy}{dt})^2 = 0 \), we follow the hint in the paragraph above the problem. After changing variables to \( v = \frac{dy}{dt} \), we arrive at the following first order equation \( \frac{dv}{dt} + t(v)^2 = 0 \). This is a separable equation. Solving the equation gives \( -1/v = -t^2/2 + c \). Solving for \( v \) and remembering \( v = \frac{dy}{dt} \), we get \( \frac{dv}{dt} = v = \frac{-2}{c_1 - t} \). We are not done yet. We still have to solve for \( y(t) \). We don’t know anything about the constant of integration \( c_1 \). As we try to solve for \( y \), we find that we need to make assumptions the sign of \( c_1 \). If \( c_1 = 0 \), then \( \frac{dv}{dt} = 2/t^2 \implies y(t) = -2/t + c_2 \). If \( c_1 > 0 \) we set \( c_1 = k^2 \). (This is a nice trick because now \( c_1 \) is clearly positive.) Solving the resulting equation by partial fractions, we get \( y(t) = \frac{1}{k} \ln |\frac{k}{k+t}| + c_2 \). If \( c_2 < 0 \), we set \( c_1 = -k^2 \). The resulting integral is an inverse tangent. Thus we get \( y(t) = \frac{2}{k} \tan^{-1}(\frac{t}{k}) + c_2 \). Lastly, notice that \( y(t) = \text{constant} \) is also a solution.

Problem 34, Page 128

Following the hint, we substitute \( v = \frac{dy}{dt} \) which gives \( y \frac{dv}{dt} + (v)^2 = 0 \). But this doesn’t really help. Now we have three variables, clearly it would be nice to get rid of one. Again following the hint we use \( \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = \frac{dv}{dy} v \) to arrive at \( y(\frac{dv}{dy})v + v^2 = 0 \). This is separable and gives \( v = c_1/y \). Next we solve \( v = \frac{dy}{dt} = c_1/y \) which is again separable giving a final answer of \( y^2 = c_1 t + c_2 \).

Problem 24, Page 138

First check that both are solutions.

\( y_1(t) = e^t \implies y_1'(t) = e^t , \ y_1'' = e^t \implies y_1'' - 2y' + y = e^t - 2e^t + e^t = 0 \) Checks!!

\( y_2(t) = te^t \implies y_2'(t) = e^t + te^t , \ y_2'' = 2e^t + te^t \implies y_2'' - 2y' + y = 2e^t + te^t - 2(e^t + te^t) + e^t = 0 \) Checks!!
Now check to see if it is a complete set of solutions. In other words, are they linearly independent. To do this we use the Wronskian.

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

$$W = e^{2t}$$

We see that $W$ is never zero for any $t$, hence they are linearly independent solutions.

**Problem 28, Page 138**

Again we take inspiration from the paragraph above the question. We see that if $P''(x) - Q'(x) + R(x) = 0$ then the equation is exact. In our case, $P = 1$, $Q = x$, $R = 1$. Plugging these in we find that the equation is exact. The point is to get the equation in the form

$$\frac{d}{dx}[P(x)\frac{dy}{dx}] + \frac{d}{dx}[f(x)y] = 0 \text{ (*)}$$

because we can integrate this once arriving at

$$P(x)\frac{dy}{dx} + f(x)y =$$

which we can solve. So, the question is what is $f(x)$ in terms of $R, Q, P$. To see this, we expand equation (*). This gives

$$lP'y' + Py'' + f'y + fy'$$

$$Py'' + (P' + f)y' + f'y = 0$$

We know that $Py'' + Qy' + R = 0$. By equating terms in the two equations we get that $Q - P' = f$. Hence in our case $f = x$. After integrating equation (*) once, we get $y' + xy = c_1$. This is just a first order ODE that we know how to solve from previous sections. We get

$$y = c_1e^{-x^2/2} \int_{0}^{x} e^{x^2/2} dx + c_2e^{-x^2/2}$$
Problem 17, Page 144

Problem  Find the Wronskian of two solutions of the given differential equation without solving the equation,

\[ x^2y'' + xy' + (x^2 - \nu^2)y = 0. \quad \text{Bessel’s equation} \quad (1) \]

Solution  Here we will use the Abel’s theorem. In this problem, \( P(x) = 1/x \) and \( Q(x) = (x^2 - \nu^2)/x^2 \). The Wronskian is given by

\[ W(y_1, y_2)(x) = c \exp\left[ - \int p(x) dx \right] \quad (2) \]

i.e.,

\[ W(x) = c \exp\left[ - \int \frac{1}{x} dx \right] = c \exp(-\ln x). \quad (3) \]

Thus, we have

\[ W(x) = \frac{c}{x}. \quad (4) \]

Problem 24, Page 150

Problem  Consider the initial value problem

\[ 5u'' + 2u' + 7u = 0, \quad u(0) = 2, \quad u'(0) = 1. \quad (5) \]

(a) Find the solution \( u(t) \) of this problem.

Solution  The characteristic equation for this problem is

\[ 5r^2 + 2r + 7 = 0. \quad (6) \]

Solving this equation, we get \( r_1 = (-1 + \sqrt{34}i)/5 \) and \( r_2 = (-1 - \sqrt{34}i)/5 \). Thus,

\[ u = c_1 \exp\left[(-1 + \sqrt{34}i)t/5\right] + c_2 \exp\left[(-1 - \sqrt{34}i)t/5\right]. \quad (7) \]
Re-organizing equation (7), we have

\[ u = (c_1 e^{\sqrt{34}t/5} + c_2 e^{-\sqrt{34}t/5})e^{-t/5}. \]  \hspace{1cm} (8)

Plugging the initial conditions into equation (8) yields

\[ u(t) = e^{-t/5}[2\cos(\sqrt{34}t/5) + 7/\sqrt{34}\sin(\sqrt{34}t/5)]. \]  \hspace{1cm} (9)

(b) Find the smallest \( T \) such that \( |u(t)| \leq 0.1 \) for all \( t > T \).

Solution \hspace{1cm} For all \( t \),

\[ |u(t)| \leq MAX(2\cos(\sqrt{34}t/5) + 7/\sqrt{34}\sin(\sqrt{34}t/5))e^{-t/5}. \]  \hspace{1cm} (10)

where

\[ MAX(2\cos(\sqrt{34}t/5) + 7/\sqrt{34}\sin(\sqrt{34}t/5)) = 2.33. \]  \hspace{1cm} (11)

Thus, the problem is to find the smallest \( T \), such that \( |2.33e^{-t/5}| \leq 0.1 \). Solving this equation, we have \( T = 15.74 \). Notice: this estimated \( T \) is larger than the smallest \( T \) that could be obtained from the plot or the calculator.

Solution 2: Matlab \hspace{1cm} see Fig. 2

Problem 39, Page 153

Problem \hspace{1cm} Use the hint for solving the Euler Equations to solve the following given equation

\[ t^2y'' + ty' + y = 0 \]  \hspace{1cm} (12)

Solution \hspace{1cm} First, we know that \( y' = \frac{1}{t}\frac{dy}{dx} \) and \( y'' = \frac{1}{t^2}(\frac{d^2y}{dx^2} - \frac{dy}{dx}) \). Substitution of \( x = \ln t \) into equation (12) yields

\[ y'' + y = 0. \]  \hspace{1cm} (13)
Figure 2: Solution for Problem 24, Page 150
The characteristic equation for the above homogeneous equation is

\[ r^2 + 1 = 0, \]

where \( r = \pm i \).

Thus,

\[ y = c_1 \cos x + c_2 \sin x = c_1 \cos(\ln t) + C_2 \sin(\ln t), \]

and \( t > 0 \).

**Problem 23, Page 161**

**Problem** Use the method of reduction of order to find a second solution of the given differential equation

\[ t^2 y'' - 4ty' + 6y = 0, \quad y_1(t) = t^2 \]  

(16)

**Solution** Assume \( y_2(t) = v(t)t^2 \), then \( y'_2 = v't^2 + 2vt \), and \( y'' = v''t^2 + 4v't + 2v \). Substitute these relations into equation (16), we have

\[ v'' = 0. \]

(17)

Then,

\[ v = c_1 t + c_2. \]

(18)

Thus, the second solution is

\[ y_2(t) = t^3. \]

(19)
**Problem 1, Page 177**

**Problem** Use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

\[ y'' - 5y' + 6y = 2e^t \]  \hspace{1cm} (20)

**Solution 1: Variation of Parameters** First, let’s take a look of the homogeneous equation

\[ y'' - 5y' + 6y = 0. \]  \hspace{1cm} (21)

The general solution for this equation is

\[ y(t) = c_1 e^{2t} + c_2 e^{3t}. \]  \hspace{1cm} (22)

Then a particular solution for the nonhomogeneous equation will have the form

\[ y = u_1(t)e^{2t} + u_2(t)e^{3t}. \]  \hspace{1cm} (23)

Differentiating equation (23) yields

\[ y' = u_1' e^{2t} + 2u_1(t)e^{2t} + u_2' e^{3t} + 3u_2(t)e^{3t}. \]  \hspace{1cm} (24)

We set the term involving \( u_1' \) and \( u_2' \) equal to zero, i.e.,

\[ u_1' e^{2t} + u_2' e^{3t} = 0, \]  \hspace{1cm} (25)

and

\[ y' = 2u_1(t)e^{2t} + 3u_2(t)e^{3t}. \]  \hspace{1cm} (26)

Then, differentiating again, we get
\[ y'' = 2u'_1 e^{2t} + 4u_1(t)e^{2t} + 3u'_2 e^{3t} + 9u_2(t)e^{3t}. \]  

(27)

Plug equation (23) (26) and (27) into equation (20), we have

\[ 2u'_1 e^{2t} + 3u'_2 e^{3t} = 2e^t. \]  

(28)

Solving equation (25) and (28), we have

\[ u_1 = -e^{-t} + c_1, \quad u_2 = 2e^{-2t} + c_2. \]  

(29)

Thus, the general solution is

\[ y(t) = c_1 e^{2t} + c_2 e^{3t} + e^t \]  

(30)

and the particular solution is

\[ Y(t) = e^t. \]  

(31)

**Solution 2: Undetermined coefficients** Since the nonhomogeneous term has the form \(e^t\), we assume

\[ Y(t) = Ae^t. \]  

(32)

Substitute \(Y(t)\) into equation (20), we get

\[ (A - 5A + 6A)e^t = 2e^t. \]  

(33)

Then, \(A = 1.\)