Page 201, Problem 25:

\[ u'' + \frac{1}{5} u' + u + \frac{1}{5} u^3 = \cos \omega t \]

\[ u(0) = 0, u'(0) = 0 \]

(a)
First write the M_file `ty.m`:

```matlab
function yprime = ty(t,y);
global omega;
yprime = [y(2);
-y(2)/5-y(1)-y(1)^3/5+cos(omega*t)];
```

In Matlab, write:

```matlab
[t,y] = ode23('ty',0,40,[0;0]);
```

with different omega, we can get:
With Matlab command “max()”, we can get R:
When $\omega=0.5$, $R=1.5261$;
When $\omega=1$, $R=2.2973$;
When $\omega=2$, $R=0.5712$.

As to the nonlinear term $\varepsilon u^3$. If we think physically, the nonlinear term will block the movement if $\varepsilon$ is positive. On the other hand, if $\varepsilon$ negative, as $u$ increases, the nonlinear term will contribute more until it is larger than the linear term. Then there will be no oscillation.

For the equation:
$$u'' + \gamma u' + ku + k_1 u^n = \cos(\omega t)$$
When $t >> \frac{2}{\gamma}$, $y_{\text{homogeneous}} \sim 0$

and $y = y_{\text{homogeneous}} + y_{\text{particular}} \sim y_{\text{particular}}$

$y_{\text{particular}}$ has no relation with the initial condition and $y$ goes to the steady state.

(b) Similarly use Matlab, we can draw the relation between $R$ and $\omega$:

![Graph showing the relation between Omega and amplitude R](image)

If the equation has no nonlinear term, we can anticipate that when $\omega=1$, $R$ gets its maximum amplitude. When we take into account the nonlinear term, we can see at $\omega=1.4$, $R$ get it’s maximum value.

(c) For the linear spring, we can solve the equation analytically:
From the figure, we can see that with linear spring, when \( \omega=1 \), the spring will be in resonance. All the energy from the forcing term will go to the spring and only the damping term will cost the energy to lose. With the nonlinear term exist, the forcing term will never give all it’s energy to the spring.

Page 208, Problem 26:

Given that \( y_1 \) is a solution of the equation

\[
y'''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0
\]

we must show that the substitution \( y = y_1(t)v(t) \) leads to an equation of second order for \( v' \).
First let’s write the expressions for \( y' \), \( y'' \), and \( y''' \)

\[
y' = y_1'v + v'y_1 \\
y'' = y_1''v + 2y_1'v' + v''y_1 \\
y''' = y_1'''v + 3y_1''v' + 3y_1'v'' + v'''y_1
\]

Now substituting these expressions into our original equation we obtain

\[
y_1''''v + 3y_1''v' + 3y_1'v'' + v'''y_1 + p_1(y_1''v + 2y_1'v' + v''y_1) + p_2(y_1'v' - p_1y_1''v' + v'''y_1) = 0
\]

The terms can be rearranged in the following way

\[
y_1v'''' + (3y_1' + p_1y_1)v'' + (3y_1'' + 2p_1y_1' + p_2y_1)v' + (y_1''' + p_1y_1'' + p_2y_1) = 0
\]

It is clear that the coefficient in front of the \( v \) term is in fact the original homogeneous ODE for \( y_1 \) and so this term vanishes and we are left with the result

\[
y_1v'''' + (3y_1' + p_1y_1)v'' + (3y_1'' + 2p_1y_1' + p_2y_1)v' = 0
\]

which is a second order ODE for \( v' \).

Page 208, Problem 27:

\[
(2-t)y^{(3)} + (2t-3)y'' - ty' + y = 0, \quad t < 2
\]

\[
y_1(t) = e^t
\]
if the problem does not give us $y_1(t)$, we can guess it. Look at the equation, the most easy way is to guess $y_1(t)=\exp(st)$. But if we guess in this way, in the characteristic equation, we only have 1 unknown $s$. We hope the coefficients before both term $t$ and 1 are equal to 0 in the characteristic equation. So we can assume $y_1(t)=\exp(st)+b$. $s$ and $b$ are unknown. Plug the solution in, we will find $y_1(t)$.

In the equation, we have got $y_1(t)=\exp(t)$. Assume:

$$y(t) = y_1(t)v(t)$$

$$y' = e^t(v + v')$$

$$y'' = e^t(v + 2v' + v'')$$

$$y''' = e^t(v + 3v' + 3v'' + v''')$$

Plug $v$ into equation, we can get:

$$(2-t)e^t(v + 3v' + 3v'' + v''') + (2t-3)e^t(v + 2v' + v'')$$

$$-te^t(v + v') + e^tv = 0$$

$$(2-t)v'' + (3-t)v'' = 0$$

Assume $u=v''$, then we have

$$(2-t)u' = -(3-t)u$$,

or

$$\frac{du}{u} = -\frac{3-t}{2-t}dt = (-1 + \frac{1}{t-2})dt$$

$$\ln|u| = -t + \ln(2-t) + c$$

$$u = c_1(2-t)e^{-t} = v''$$

$$v' = \int c_1(2-t)e^{-t}dt = c_1(-e^{-t} + te^{-t} + c_2)$$

$$v = c_1te^{-t} + c_2t + c_3$$

So we can get:
\[ y(t) = y_1(t)v(t) = c_1 t + c_2 te^t + c_3 e^t \]

Page 214 Problem 17:

\[ y^{vi} - 3y^{iv} + 3y'' - y = 0 \]

The characteristic equation for this problem is

\[ r^6 - 3r^4 + 3r^2 - 1 = \left(r^2 - 1\right)^3 = 0 \]

which has roots, \( r=-1,1, \) both with a multiplicity of three.

Each root therefore contributes three terms to the general solution, \( \exp(r\,t), t\,\exp(r\,t) \) and \( t^2\,\exp(r\,t) \). The general solution is written

\[ y = c_1 \exp(t) + c_2 t\exp(t) + c_3 t^2 \exp(t) + c_4 \exp(-t) + c_5 t \exp(-t) + c_6 t^2 \]

Page 214, Problem 29:

\[ y''' + y' = 0; \ y(0)=0, \ y'(0)=1, \ y''(0)=2 \]

This is third order ODE, we suppose we can get 3 integration constant in the general solution. This is constant coefficient ODE, with 3 boundary conditions, there should be no problem that we get 3 integration constant.

Assume: \( y=\exp(st) \), we can get characteristic equation:

\[ n^3 + n = 0 \]

\[ n_1 = 0; n_2 = i; n_3 = -i \]
The three linear independent solutions are:

\[ y_1 = 1 \]
\[ y_2 = \cos(t) + \sin(t) \]
\[ y_3 = \cos(t) - \sin(t) \]

So the general solution is:

\[ y = c_1 + c_4 \cos(t) \sin(t) + c_5 \cos(t) - \sin(t) \]

\[ = c_1 + c_2 \cos(t) + c_3 \sin(t) \]

We get the constant c’s, plug the general solution into boundary condition:

\[ y(0) = 0 \Rightarrow c_1 + c_2 = 0 \]
\[ y'(0) = 1 \Rightarrow c_3 = 1 \]
\[ y''(0) = 2 \Rightarrow -c_2 = 2 \]

\[ y = 2 - 2 \cos(t) + \sin(t) \]

To prove this is the solution, plug it into the ODE and check it. You can also check the boundary condition.

We can use Matlab, or KaleidaGraph, or Excel etc to draw the solution:
We can see as t goes to infinity, y will continue to oscillate.

Page 215 Problem 39:

The spring-mass system is shown below
(a) First we going to derive the equations of motion of this system, starting from Newton’s law of motion, \( F = ma \). The force exerted by a spring fixed at one end is \( F = -k\Delta L \), where \( \Delta L \) is the displacement from the equilibrium position. The equilibrium positions of \( m_1 \) and \( m_2 \) are shown in the figure and positive displacements are in the downward direction as indicated.

\( m_1 \) feels a force both from spring 1 and spring 2 and so its equation of motion involves the sum of both of these forces
\( m_1 a_1 = -k_1 \Delta L_1 - k_2 \Delta L_2 \)

In this equation we must identify the terms \( \Delta L_1 \), \( \Delta L_2 \) and \( a_1 \). Since spring 1 is fixed we can apply \( F = -k \Delta L \) and find that a positive displacement, \( u_1 \), gives rise to a force in the direction opposite to \( u_1 \) and so \( \Delta L_1 = -u_1 \). Unlike spring 1, both of the endpoints of spring 2 are free to move so the force on \( m_1 \) from spring 2 depends on both \( u_1 \) and \( u_2 \). Because of the linear behavior of a spring we can consider an arbitrary movement of spring 2 as the superposition of a displacement, \( u_1 \), with the endpoint at \( m_2 \) fixed and a displacement, \( u_2 \), with the endpoint at \( m_1 \) fixed. In the first case a force, \( k_2 u_1 \), acts on \( m_1 \) in the direction opposite to \( u_1 \). In the second case a force, \( k_2 u_2 \), acts on \( m_1 \) in the direction parallel to \( u_2 \). Thus the total force acting on \( m_1 \) from an arbitrary movement of spring 2 is \(-k_2 (u_1-u_2)\) and so \( \Delta L_2 = (u_1 - u_2) \). To complete the picture the acceleration of \( m_1 \) is expressed as \( a_1 = u_1'' \).

\( m_2 \) feels a force from spring 2 only. This force is equal and opposite to the force spring 2 exerts on \( m_1 \).

\( m_2 a_2 = k_2 \Delta L_2 \)

where \( a_2 = u_2'' \).

Using the above expressions and recalling the values for \( m_1 \), \( k_1 \) and \( k_2 \) we find equations for \( u_1 \) and \( u_2 \).
\( u_1'' = -3u_1 - 2(u_1 - u_2) \Rightarrow u_1'' + 5u_1 = 2u_2 \)

\( u_2'' = 2(u_1 - u_2) \Rightarrow u_2'' + 2u_2 = 2u_1 \)

(b) Solving for \( u_2 \) in the equation for \( u_1'' \) and substituting into the equation for \( u_2'' \) we obtain

\( u_1^{iv} + 7u_1'' + 6u_1 = 0 \)

The characteristic equation for this fourth order ODE is

\( r^4 + 7r^2 + 6 = 0 \)

which has roots \( r = \pm i, \pm i\sqrt{6} \). The roots are complex, and each complex conjugate pair contributes a term to the general solution of the form \( \cos(rt) + \sin(rt) \). The general solution is

\( u_1 = c_1 \sin(t) + c_2 \cos(t) + c_3 \sin(\sqrt{6}t) + c_4 \cos(\sqrt{6}t) \)

(c) The initial conditions for the spring-mass system are given as \( u_1(0)=1, \ u_1'(0)=0, \ u_2(0)=2, \ u_2'(0)=0 \). Using the equation, \( u_1'' + 5u_1 = 2u_2 \), we find that \( u_1''(0) = -1 \) and taking the derivative of this equation we find that \( u_1'''(0) = 0 \). Since we now have all the constants necessary to specify the initial state of \( u_1 \) we proceed to solve for \( u_1 \) using the general solution above.
\[ u_1' = c_1 \cos(t) - c_2 \sin(t) + c_3 \sqrt{6} \cos(\sqrt{6}t) - c_4 \sqrt{6} \sin(\sqrt{6}t) \]

\[ u_1'' = -c_1 \sin(t) - c_2 \cos(t) - c_3 \sqrt{6} \sin(\sqrt{6}t) - c_4 \sqrt{6} \cos(\sqrt{6}t) \]

\[ u_1''' = -c_1 \cos(t) + c_2 \sin(t) - c_3 \sqrt{6} \cos(\sqrt{6}t) + c_4 \sqrt{6} \sqrt{6} \sin(\sqrt{6}t) \]

\[ u_1(0) = c_2 + c_4 = 1 \]

\[ u_1'(0) = c_1 + c_3 \sqrt{6} = 0 \]

\[ u_1''(0) = -c_2 - c_4 \sqrt{6} = -1 \]

\[ u_1'''(0) = -c_1 - c_3 \sqrt{6} \sqrt{6} = 0 \]

\[ \Rightarrow c_1 = c_3 = c_4 = 0, c_2 = 1 \]

\[ \Rightarrow u_1 = \cos(t) \]

With this solution for \( u_1 \) we can find \( u_2 \).
\[ u_2'' + 2u_2 = 2\cos(t) \]

\[ u_2 = u_{2h} + u_{2p} \]

\[ u_{2h} = b_1 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t) \]

\[ u_{2p} = b_3 \cos(t) + b_4 \sin(t) \]

\[ u_{2p}' = -b_3 \sin(t) + b_4 \cos(t) \]

\[ u_{2p}'' = -b_3 \cos(t) - b_4 \sin(t) \]

\[ u_{2p}'' + 2u_{2p} = b_3 \cos(t) + b_4 \sin(t) = 2\cos(t) \]

\[ \Rightarrow u_{2p} = 2\cos(t) \]

\[ u_2 = b_1 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t) + 2\cos(t) \]

\[ u_2(0) = b_1 + 2 = 2 \]

\[ u_2'(0) = \sqrt{2}b_2 = 0 \]

\[ \Rightarrow u_2 = 2\cos(t) \]

(d) The initial conditions are now \( u_1(0) = -2, \quad u_1'(0) = 0, \]
\( u_2(0) = 1, \quad u_2'(0) = 0. \) As before we use the equation for \( u_1 \)

to obtain \( u_1''(0) \) and \( u_1'''(0). \) These values are
determined to be \( u_1''(0) = 12, \quad u_1'''(0) = 0. \) Using the same
general solution as before, the constants are now related
according to the equations
\[ u_{1}(0) = c_2 + c_4 = -2 \]

\[ u_{1}'(0) = c_1 + c_3 \sqrt{6} = 0 \]

\[ u_{1}''(0) = -c_2 - c_4 \cdot 6 = 12 \]

\[ u_{1}'''(0) = -c_1 - c_3 \cdot 6 \sqrt{6} = 0 \]

\[ \Rightarrow c_1 = c_3 = c_2 = 0, c_4 = -2 \]

\[ \Rightarrow u_{1} = -2 \cos(\sqrt{6}t) \]

We can find \( u_2 \)
$u_2'' + 2u_2 = -4\cos(\sqrt{6}t)$

$u_2 = u_2^h + u_2^p$

$u_2^h = b_1 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t)$

$u_2^p = b_3 \cos(\sqrt{6}t) + b_4 \sin(\sqrt{6}t)$

$u_2^p' = -b_3 \sqrt{6} \sin(\sqrt{6}t) + b_4 \sqrt{6} \cos(\sqrt{6}t)$

$u_2^p'' = -b_3 \sqrt{6} \cos(\sqrt{6}t) - b_4 \sqrt{6} \sin(\sqrt{6}t)$

$u_2^p'' + 2u_2^p = -4b_3 \cos(\sqrt{6}t) + 4b_4 \sin(\sqrt{6}t) = -4\cos(\sqrt{6}t)$

$\Rightarrow u_2^p = \cos(\sqrt{6}t)$

$u_2 = b_1 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t) + \cos(\sqrt{6}t)$

$u_2(0) = b_1 + 1 = 1$

$u_2'(0) = \sqrt{2}b_2 = 0$

$\Rightarrow u_2 = \cos(\sqrt{6}t)$

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Page 219, Problem 10:

$y'''' + 2y''' + y = 3t + 4$

$y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$

First calculate the general solution of homogeneous equation:

$y'''' + 2y''' + y = 0$
Assume $y=\exp(st)$, we can get characteristic equation:
\[ s^4 + 2s^2 + 1 = 0 \]
\[ s_1 = s_2 = i \]
\[ s_3 = s_4 = -i \]
We have double roots of the characteristic equation.
\[ y_h = c_1 (\cos t + i \sin t) + c_2 t (\cos t + i \sin t) \]
\[ + c_3 (\cos t - i \sin t) + c_4 t (\cos t - i \sin t) \]
\[ = (c_1 + c_2 t) \cos t + (c_3 + c_4 t) \sin t \]

To get the special solution, looking at the forcing term $3t+4$, Assume:
\[ y_s = at + b \]
Plug in, we can get $a=3$, $b=4$
So
\[ y = (c_1 + c_2 t) \cos t + (c_3 + c_4 t) \sin t + 3t + 4 \]
\[ y' = (c_2 + c_3 + c_4 t) \cos t + (-c_1 + c_4 - c_2 t) \sin t + 3 \]
\[ y'' = (-c_1 + 2c_4 - c_2 t) \cos t - (2c_2 + c_3 + c_4 t) \sin t \]
\[ y''' = -(3c_2 + c_3 + c_4 t) \cos t + (c_1 - 3c_4 + c_2 t) \sin t \]
Plug into boundary condition:
\[ y(0) = 0 \implies c_1 + 4 = 0 \]
\[ y'(0) = 0 \implies c_2 + c_3 = -3 \]
\[ y''(0) = 1 \implies -c_1 + 2c_4 = 1 \]
\[ y'''(0) = 1 \implies -3c_2 + c_3 = 1 \]
\[ \implies c_1 = -4; c_2 = 1; c_3 = -4; c_4 = -3/2 \]
So:
\[ y = (-4+t) \cos t - (4+3t/2) \sin t + (3t+4) \]
Page 224 Problem 2:

\[ y''' - y' = t \]

First let’s find the solution to the homogeneous equation, \( y''' - y' = 0 \), which is \( y = c_1 + c_2 \exp(t) + c_3 \exp(-t) \).

The variation of parameters method involves assuming a solution of the form, \( y = u_1 + u_2 \exp(t) + u_3 \exp(-t) \), where \( u_1, u_2 \) and \( u_3 \) are unknown functions.
\[ y = u_1 + u_2 \exp(t) + u_3 \exp(-t) \]

\[ y' = u_1' + u_2' \exp(t) + u_2 \exp(t) + u_3' \exp(-t) - u_3 \exp(-t) \]

Let us impose the following condition on \( u_1' \) and \( u_2' \)

\[ u_1' + u_2' \exp(t) + u_3' \exp(-t) = 0 \]

With this \( y' \) becomes

\[ y' = u_2 \exp(t) - u_3 \exp(-t) \]

Continuing this process

\[ y'' = u_2' \exp(t) + u_2 \exp(t) - u_3' \exp(-t) + u_3 \exp(-t) \]

\[ u_2' \exp(t) - u_3' \exp(-t) = 0 \]

\[ y'' = u_2 \exp(t) + u_3 \exp(-t) \]

\[ y''' = u_2' \exp(t) + u_2 \exp(t) - u_3 \exp(-t) + u_3' \exp(-t) \]

\[ y''' - y' = t \]

\[ u_2' \exp(t) + u_3' \exp(-t) = t \]

The end result is a system of equations for \( u_1' \), \( u_2' \) and \( u_3' \)
\[u_1' + u_2' \exp(t) + u_3' \exp(-t) = 0\]

\[u_2' \exp(t) - u_3' \exp(-t) = 0\]

\[u_2' \exp(t) + u_3' \exp(-t) = t\]

with solution

\[u_2' = \frac{t}{2} \exp(-t)\]

\[u_3' = \frac{t}{2} \exp(t)\]

\[u_1' = -t\]

\[u_2 = \frac{-1}{2} \exp(-t) - \frac{t}{2} \exp(-t) + c_1\]

\[u_3 = \frac{-1}{2} \exp(t) + \frac{t}{2} \exp(t) + c_2\]

\[u_1 = -\frac{t^2}{2} + c_3\]

This gives for \(y\)

\[y = -\frac{t^2}{2} + c_2 \exp(-t) + c_1 \exp(t) + K\]

where \(c_1\), \(c_2\) and \(K\) are arbitrary constants.