MAE 305
Engineering Mathematics 1
Midterm Examination 1

1. dy/dx=f(x,y)

(x,y) belong to [a,b;c,d]
We have already know that f(x,y) is continuous in [a,b;c,d]. According to Lipschitz Theorem, for any (x,y) belong to [a,b;c,d], when

$$| f(x, y_1) - f(x,y_2) | \le K | y_1 - y_2 |$$

we can be sure the solution exists and is unique.

Also according to Theorem 2.4.1 in p41 in Boyce&DiPrima, if $\partial f / \partial y$ continuous in [a,b;c,d], they solution is exist and unique.

Actually, we can prove that the two theorem are equal.

2.

a) $\dfrac{dy}{dt} + p(t)y = q(t)$

we want to find a function u(t) that makes:
uq=uy'+upy=(uy)'=uy'+u'y
so: u'=up

$u = \exp(\int pdt)$ The constant is not important here, because we just want to find a u who satisfy above equation.
So

$$uy = \int uqdt + c$$

$$y = \frac{\int uqdt + c}{u} = \frac{\int e^{\int pdt} qdt + c}{e^{\int pdt}}$$

This is 1st order ODE and we can only have 1 constant

b)

$$\frac{dy}{dx} = \frac{y\cos x + 2x\exp(y)}{1 - \sin x - x^2\exp(y)}$$

$$(y\text{con}x + 2xe^{y})dx + (-1 + \sin x + x^2 e^{y})dy = 0$$

$$M = y\cos x + 2xe^{y}$$

$$N = -1 + \sin x + x^2 e^{y}$$

$$\text{Because} M_y = \sin x + 2xe^{y} = N_x$$

This is an exact differentiation and we can find a function

$$\varphi = c$$

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = Mdx + Ndy$$

$$\varphi_x = M = y\cos x + 2xe^{y}$$

$$\varphi = y\sin x + x^2 e^{y} + h(y)$$

$$\varphi_y = N = -1 + \sin x + x^2 e^{y} = \sin x + x^2 e^{y} + h'$$

$$h(y)' = -1$$

$$h = -y + c$$

$$\varphi = y\sin x + x^2 e^{y} - y + c_1 = c$$

c)

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 6y = 2$$

This is Eular Equation, the trick is assume t=lnx, so:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{1}{x}\frac{dy}{dt}$$

$$\frac{dy}{dx}=\frac{d}{dt}(\frac{dy}{dx})\frac{dt}{dx}=\frac{d}{dt}(\frac{1}{x}\frac{dy}{dt})\frac{1}{x}=(-\frac{1}{x^2}\frac{dx}{dt}\frac{dy}{dt}+\frac{1}{x}\frac{d^2y}{dt^2})\frac{1}{x}$$

$$=\frac{1}{x^2}\frac{d^2y}{dt^2}-\frac{1}{x}\frac{dy}{dt}$$

We can simplify the equation to:
y’’(t)-5y’(t)-6y(t)=2
first solve the homogeneous equation y’’-5y’-6y=0, we can get

$$y_h=c_1\exp(6t)+c_2\exp(-t)=c_1x^6+c_2/x$$

To solve inhomogeneous solution y’’-5y’-6y=2, Assume y=a
them a=-1/3.
so the solution is:

$$y=y_h+y_p=c_1x^6+c_2/x-1/3$$

Another method to solve Eular equation is simply assume $y=x^n$
Substitute them into the equation and get n. But you are lucky this time because n1 and n2 are different. If n1=n2, what can you do?

d)

$$\frac{dy}{dx}=\frac{y^2+2xy}{x^2}=(\frac{y}{x})^2+2(\frac{y}{x})$$

This is homogeneous equation. The trick is assume:
v(x)=y/x. So
dy/dx=d(xv)/dx=v+xv’
And we can get:

$$v+xv'=v^2+2v$$

$$x\frac{dv}{dx}=v^2+v$$

$$\frac{dx}{x}=\frac{dv}{v^2+v}$$

e)

$$\frac{dy^2}{d^2x}=-(\frac{dy}{dx})^2+y^2$$

y’’=f(y’,y), this is autonomous equation. The trick is assume:
v=dy/dx

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

$$v\frac{dv}{dy} = -v^2 + y^2$$

Now we get v as a function of y. This is an 1st order ODE.

3.

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} -5 & 3 \\ 3 & 5 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{at}$$

a) This is a system of first order linear differential equations. First we find the solution to the corresponding homogeneous problem.

$$\frac{d\vec{x}_h}{dt} = \begin{bmatrix} -5 & 3 \\ 3 & 5 \end{bmatrix}\vec{x}_h$$

This involves finding the eigenvalues of the matrix $\begin{bmatrix} -5 & 3 \\ 3 & 5 \end{bmatrix}$.

$$\mathrm{Det}\begin{bmatrix} -5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} = (-5-\lambda)(5-\lambda) - 9 = 0$$

$$\lambda_1 = \sqrt{34}$$

$$\lambda_2 = -\sqrt{34}$$

b) Now we find eigenvectors corresponding to these eigenvalues.

For $\lambda_1 = \sqrt{34}$ we have to find a vector $\vec{\varepsilon}_1$ such that

$$\begin{bmatrix} -5-\lambda_1 & 3 \\ 3 & 5-\lambda_1 \end{bmatrix}\vec{\varepsilon}_1 = 0$$

$$\begin{bmatrix} \left(-5-\sqrt{34}\right) & 3 \end{bmatrix}\vec{\varepsilon}_1 = 0$$

$$\vec{\varepsilon}_1 = \begin{bmatrix} 1 \\ \dfrac{5+\sqrt{34}}{3} \end{bmatrix}$$

For $\lambda_2 = -\sqrt{34}$

$$\begin{bmatrix} -5-\lambda_2 & 3 \\ 3 & 5-\lambda_2 \end{bmatrix}\vec{\varepsilon}_2 = 0$$

$$\begin{bmatrix} \left(-5+\sqrt{34}\right) & 3 \end{bmatrix}\vec{\varepsilon}_2 = 0$$

$$\vec{\varepsilon}_2 = \begin{bmatrix} 1 \\ \dfrac{5-\sqrt{34}}{3} \end{bmatrix}$$

c) Having found the eigenvalues and a set of corresponding eigenvectors we can write the homogeneous solution as

$$\vec{x}_h = c_1\vec{\varepsilon}_1 e^{\sqrt{34}t} + c_2\vec{\varepsilon}_2 e^{-\sqrt{34}t}$$

d) The fundamental matrix is formed by placing the two linearly independent solutions into the columns of a matrix.

$$\Psi = \begin{bmatrix} e^{\sqrt{34}t} & e^{-\sqrt{34}t} \\ \dfrac{\left(5+\sqrt{34}\right)e^{\sqrt{34}t}}{3} & \dfrac{\left(5-\sqrt{34}\right)e^{-\sqrt{34}t}}{3} \end{bmatrix}$$

e) Using the fundamental matrix the inhomogeneous equation can be solved by the variation of parameters method. We assume a solution of the form, $\vec{\mathbf{x}} = \Psi\vec{\mathbf{u}}$, and after substituting this into the differential equation we obtain,

$$\Psi\vec{\mathbf{u}}' = \begin{bmatrix}1\\2\end{bmatrix}e^{at}$$

$$\vec{\mathbf{u}}' = \begin{bmatrix}u_1{}'\\u_2{}'\end{bmatrix}$$

$$u_1{}'e^{\sqrt{34}t} + u_2{}'e^{-\sqrt{34}t} = e^{at}$$

$$u_1{}'\frac{\left(5+\sqrt{34}\right)}{3}e^{\sqrt{34}t} + u_2{}'\frac{\left(5-\sqrt{34}\right)}{3}e^{-\sqrt{34}t} = 2e^{at}$$

$$u_1{}' = -u_2{}'\frac{\left(5-\sqrt{34}\right)}{\left(5+\sqrt{34}\right)}e^{-2\sqrt{34}t} + \frac{6}{\left(5+\sqrt{34}\right)}e^{\left(a-\sqrt{34}\right)t}$$

$$u_2{}'\frac{2\sqrt{34}}{\left(5+\sqrt{34}\right)}e^{-\sqrt{34}t} + \frac{6}{\left(5+\sqrt{34}\right)}e^{at} = e^{at}$$

$$u_2{}' = \frac{\left(-1+\sqrt{34}\right)}{2\sqrt{34}}e^{\left(a+\sqrt{34}\right)t}$$

$$u_1{}' e^{\sqrt{34}t} + u_2{}' e^{-\sqrt{34}t} = e^{at}$$

$$u_2{}' = \frac{\left(-1+\sqrt{34}\right)}{2\sqrt{34}} e^{\left(a+\sqrt{34}\right)t}$$

$$u_1{}' = \frac{\left(1+\sqrt{34}\right)}{2\sqrt{34}} e^{\left(a-\sqrt{34}\right)t}$$

The above equations can be integrated to find u1 and u2. First we will consider the case where a is not equal to either of the two eigenvalues.

$$u_1 = \frac{\left(1+\sqrt{34}\right)}{2\sqrt{34}\left(a-\sqrt{34}\right)} e^{\left(a-\sqrt{34}\right)t} + c_1$$

$$u_2 = \frac{\left(-1+\sqrt{34}\right)}{2\sqrt{34}\left(a+\sqrt{34}\right)} e^{\left(a+\sqrt{34}\right)t} + c_2$$

Thus the solution to the inhomogeneous equation is

$$\vec{x} = \Psi\vec{u} = \begin{bmatrix} e^{\sqrt{34}t} & e^{-\sqrt{34}t} \\ \frac{\left(5+\sqrt{34}\right)e^{\sqrt{34}t}}{3} & \frac{\left(5-\sqrt{34}\right)e^{-\sqrt{34}t}}{3} \end{bmatrix} \begin{bmatrix} \frac{\left(1+\sqrt{34}\right)}{2\sqrt{34}\left(a-\sqrt{34}\right)} e^{\left(a-\sqrt{34}\right)t} + c_1 \\ \frac{\left(-1+\sqrt{34}\right)}{2\sqrt{34}\left(a+\sqrt{34}\right)} e^{\left(a+\sqrt{34}\right)t} + c_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \frac{(1+a)}{\left(a^2-34\right)} \\ \frac{(13+2a)}{\left(a^2-34\right)} \end{bmatrix} e^{at} + c_1\vec{\varepsilon}_1 e^{\sqrt{34}t} + c_2\vec{\varepsilon}_2 e^{-\sqrt{34}t}$$

f) If a is equal to one of the eigenvalues, the method of variation of parameters still works. In fact as long as the solution to the homogeneous problem is known, variation of parameters can be used to find the solution for the inhomogeneous equation. The solution of this problem, using undetermined coefficients, involves finding a particular solution to the differential equation, xp. This is accomplished by assuming that

$\vec{x}_p = \vec{a}te^{at} + \vec{b}e^{at}$, where the vectors a and b are determined by substituting xp into

the differential equation. The solution to the inhomongeneous problem is then
$\vec{x} = \vec{x}_h + \vec{x}_p$.