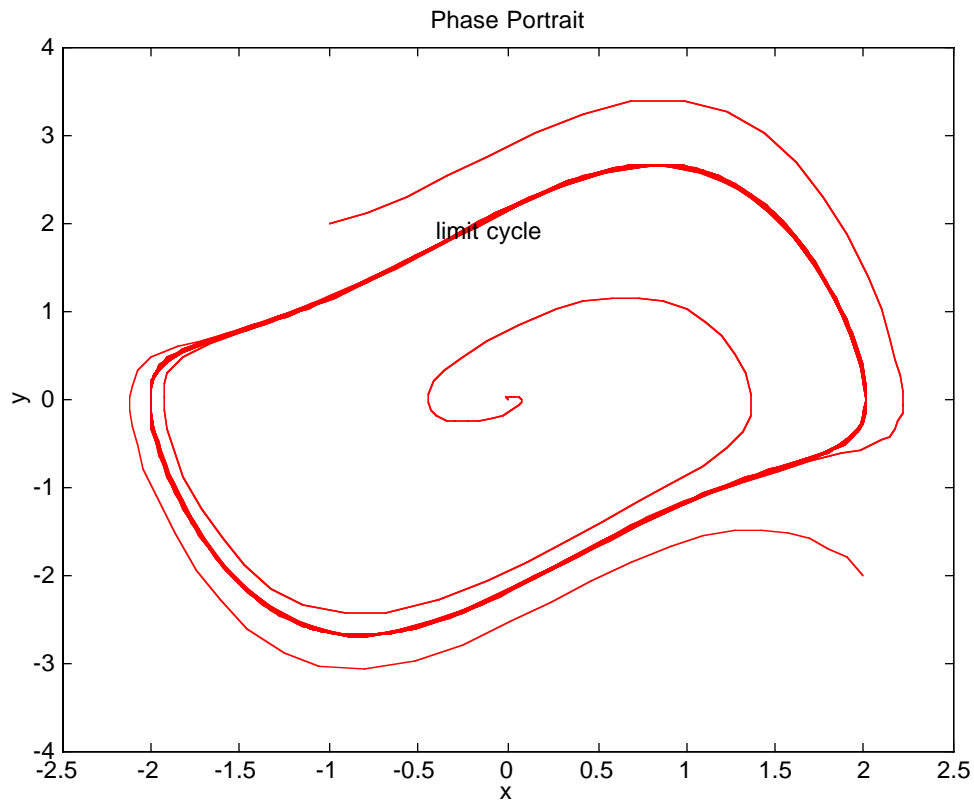


**MAE 305**  
**Engineering Mathematics I**  
**Answer to Second Mid-Term**

**1.**

a) The singular point at  $(x,y)=(0,0)$ .



There exists a limit cycle, the solutions that start from outside the limit cycle will go inward to approach the limit cycle, the solutions that start from inside the limit cycle will spiral outward to approach the limit cycle.

The solutions that were started at points far, far away from the origin would spiral inward and go to the limit cycle.

b) The singular point  $(x,y)=(0,0)$  is an unstable spiral point.

c) The linear equation as following

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 - 2xy & 1 - x^2 \end{pmatrix}_{(0,0)} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues are  $\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$   
 so the singular point  $(x,y)=(0,0)$  is an unstable spiral point.

d) Eventually, all trajectories of solutions which were started at some arbitrary points (except precisely at the singular points) will approach the limit cycle.

## 2. Laplace Transform.

a)

$$\begin{aligned} L\left\{\frac{dy}{dt}\right\} &= \int_0^{\infty} \frac{dy}{dt} e^{-st} dt \\ &= \int_0^{\infty} e^{-st} dy = ye^{-st} \Big|_0^{\infty} - \int_0^{\infty} y(-s)e^{-st} dt \\ &= -y(0) + s \int_0^{\infty} ye^{-st} dt \\ &= sY(s) - y(0) \end{aligned}$$

b)

$$\begin{aligned} L\{u_c(t)\} &= \int_0^{\infty} u_c(t)e^{-st} dt = \left(\int_0^c + \int_c^{\infty}\right) u_c(t)e^{-st} dt \\ &= 0 + \int_c^{\infty} e^{-st} dt \\ &= -\frac{1}{s} e^{-st} \Big|_c^{\infty} \\ &= \frac{1}{s} e^{-cs} \end{aligned}$$

3. Solve the following ODE by Laplace Transform:

$$y'' + 3y' + 2y = \delta(t - 5) \quad y(0) = 0, \quad y'(0) = 0$$

**Answer:**

$$\begin{aligned} L(y'' + 3y' + 2y) &= s^2 Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y \\ &= (s^2 + 3s + 2)Y \end{aligned}$$

$$L(\delta(t - 5)) = e^{-5s}$$

Thus, we have the transformed equation as

$$Y = \frac{e^{-5s}}{s^2 + 3s + 2} = e^{-5s} \left( \frac{1}{s+1} - \frac{1}{s+2} \right).$$

According to the table,

$$L^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c),$$

we have

$$y = u_5(t)\{e^{-(t-5)} - e^{-2(t-5)}\}.$$

4. Obtaining the series solution about  $x=0$  of the following ODE

$$xy'' + y = 0.$$

**Answer:**

Let  $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ , we get

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Let  $m=n-1$ , then for the first term we have

$$\sum_{m=-1}^{\infty} (m+r+1)(m+r)a_{m+1} x^{m+r} = r(r-1)a_0 x^{r-1} + \sum_{m=0}^{\infty} (m+r+1)(m+r)a_{m+1} x^{m+r}.$$

Thus,  $r(r-1)=0$ . This answers the first question. Also, we have

$$\sum_{n=-1}^{\infty} [(n+r+1)(n+r)a_{n+1} + a_n] x^{n+r} = 0.$$

The recurrence relation is

$$a_{n+1} = \frac{-a_n}{(n+r+1)(n+r)}.$$

We can only use this equation to get the serious solution for the larger  $r$ , since in this problem  $r_1 - r_2$  is a integer.

Replacing  $r=1$  into the above equation, one get

$$a_{n+1} = \frac{-a_n}{(n+2)(n+1)}.$$

The second solution can not be easily solved by repeating the above steps.