MAE 305
Engineering Mathematics I
Princeton University

Final Exam, Closed-book, 2 hours and 50 minutes

January 21, 1997

It is a good idea to first look through all the questions, and do the easy ones first, the harder ones later. Whenever appropriate, always explains in English what you are doing. If you need some formula and you don't remember what it is exactly, tell me what you need, and proceed symbolically, explaining as you go.

The maximum grade point you can get is 200.

1. The one-dimensional unsteady **Heat Conduction** PDE for $u(x,t)$ is

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where

$$\alpha(t) = 1 + e^{-t}. \tag{2}$$

It is important to note that α is always positive. The region of interest is $0 \leq x \leq 1$ and $t \geq 0$. The boundary conditions are:

$$\frac{\partial u}{\partial x}(0,t) = 0, \tag{3}$$

$$\frac{\partial u}{\partial x}(1,t) = 0. \tag{4}$$

The initial condition is:

$$u(x,0) = f(x) \tag{5}$$

where $f(x)$ is a known given function of x. Solve this problem by separation of variables. The solution is expressed as

$$u = \sum_{n=0}^{\infty} A_n X_n(x) T_n(t) \tag{6}$$

(a) Find the ODE's for the $X_n(x)$'s and the $T_n(t)$'s. Solve them. (20 points)

(b) Determine the separation constants. (10 points)

(c) Determine the coefficients A_n's (express your answer in a form ready for numerical computation) (15 points)

(d) For $f(x) = \delta(x - \frac{1}{2})$ where $\delta(x - x_o)$ is the *Dirac Delta Function*, evaluate A_0 and A_3. (5 points)

2. Here is a linear differential operator:

$$L \equiv (1 + x^2)\frac{d}{dx} + 2x \tag{7}$$

(a) Solve the ODE problem $L(u) = 1$ with initial condition $u(0) = 0$. (10 points)

(b) The region of interest is $0 \le x \le 1$. Find the adjoint operator L^* associated with this L. Is L a self-adjoint operator? (15 points)

(c) Say something intelligent about the term that is to be evaluated at $x = 0$ and $x = 1$. (5 points)

3. Here is a two variables ODE system:

$$\frac{dx}{dt} = x - y^2, \qquad \frac{dy}{dt} = y - x^2 \tag{8}$$

a. Draw the phase portrait, and identify the singular points (how many, and where). I want arrows on the trajectories indicating the movement for increasing time. For each singular point, identify what kind it is. (20 points)

b. Pick one singular point (your choice), and derived the appropriate linear equations (under a "microscope," using my metaphor) valid in the neighborhood of that singular point. Do an eigen-analysis there to confirm your conclusions obtained by inspection of the phase portrait. (10 points)

4. Solve the following ODE by Laplace Transform:

$$y'' + y = u_{3\pi}(t), \qquad y(0) = 1, \qquad y'(0) = 0, \tag{9}$$

where $u_{3\pi}(t)$ is the *step function*. Use the Table provided. (20 points)

5. Given the following two variables ODE

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 9 \\ -1 & -5 \end{bmatrix} \mathbf{x} \tag{10}$$

where $\mathbf{x}$ is a two-dimensional column vector.

(a) Verify that this matrix has a repeated eigenvalue. Denote this eigenvalue by r. (5 points)

(b) Find one solution to this problem. Denote this solution by $\mathbf{x} = \boldsymbol{\xi}\ e^{rt}$. In other words, find the vector $\boldsymbol{\xi}$. (10 points)

(c) Use the method of undetermined coefficients to find another solution which is linearly independent from the first one you found. The "inspirational guess" for this solution is $(\boldsymbol{\xi}\ \mathrm{t} + \boldsymbol{\eta})\ e^{rt}$. In other words, use the provided inspirational guess, find the vector equation that $\boldsymbol{\eta}$ must satisfy. Then find the vector $\boldsymbol{\eta}$ if you have time. (40 points)

6. Answer the following questions succinctly.

a. What is a quasi-linear ODE? Give an example of an ODE which is not quasi-linear. (5 points)

b. For the ODE given in the following form:

$$\frac{dy}{dx} = f(x, y), \tag{11}$$

what are the sufficient conditions for existence and uniqueness of solution in the neighborhood of the point x_o, y_o? (5 points)

c. What did we learn when we studied the Clairnaut equation? (5 points)

Good luck!