MAE 305
Engineering Mathematics I
Princeton University

Tips on the Second Mid-Term to be held on

November 26, 1997

PMI lecture room, 11:00AM

1. **The Second Mid-Term** covers Chapter 9 (nonlinear differential equations and stability), Chapter 6 (Laplace Transform), and Chapter 5 (Series Solutions). Materials previously covered in Mid-Term #1 are assumed known (*e.g.* direction fields, undetermined coefficients, integration factor, ...).

2. **Chapter 9, Nonlinear differential equations and stability**. This is the fun chapter. The basic ideas are presumably good for more than two variables, but the most successful applications of the basic ideas are for two variable problems. (Remember, there is no chaos in two-dimensional problems).

   - What does "autonomous" mean? Do you know how to make an nonautonomous problem into an autonomous problem? (ask me in class).
   - What is meant by "stability" of an equilibrium point? What is asymptotic stability?
   - Given a two-dimensional problem $dx/dt = F(x, y)$, and $dy/dt = G(x, y)$. You should be be pretty comfortable at sketching direction fields (called "phase plane" sketch here). Use the trick I taught you (make sure you know whether the sign of F (or G) changes (or does not change) across the $F = 0$ (or $G = 0$) line(s)).
   - Once you have located the "crossing points" of the $F = 0$ line(s) with the $G = 0$ line(s), you have located the singular points. (Note: sometimes $F = 0$ has more than one lines, and the crossing points among themselves are not singular points. The same

remark applies to $G = 0$). The phase plane graphical sketch can give you a very good hint on what kind of singular point(s) you are dealing with. To pin your conclusion down, you need to do more work.

- How do you study the properties of a singular point? You look at it under a microscope, of course! Mathematically, this means you expand $F(x, y)$ and $G(x, y)$ about the singular point (say x_o, y_o) using a Taylor Series expansion, keeping only the lowest order terms (because $x - x_o$ and $y - y_o$ are very small).

- The Taylor series of F(x,y) about x_o, y_o is:

$$\begin{aligned} F(x,y) &= F(x_o, y_o) && (1)\\ &+ \left(\frac{\partial F}{\partial x}\right)_{x=x_o, y=y_o} (x - x_o) && (2)\\ &+ \left(\frac{\partial F}{\partial y}\right)_{x=x_o, y=y_o} (y - y_o) + \ldots && (3) \end{aligned}$$

  If x_o, y_o is a singular point, the first term is zero. The terms represented by $\ldots$ are "higher order" and are negligible (if $x - x_o$ is small, then $(x - x_o)^2$ is smaller!). In other words: *any smooth curve under a microscope looks like a straight line*!

- Note: Often, you don't need to compute all those derivatives to get your Taylor Series. For example, synthetic division is often an easier way to do things (what is the Taylor series of $1/(1 - x)$ about $x = 0.2$?).

- So, any 2-dimensional singular point under a microscope can be looked at in the phase plane:

$$\frac{d\eta}{d\xi} = \frac{a\xi + b\eta + \ldots}{c\xi + e\eta + \ldots} \tag{4}$$

  where a, b, c, e are constants and the $\ldots$ are terms we are neglecting under the microscope. You are expected to be able to fully discuss a singular point with given values of a, b, c, e, using information about the eigenvalues and eigenvectors.

- With the ... terms neglected, you can make statements about the stability of the original system using the local linearized analysis. Your linear conclusions are applicable to the case when the ... terms are not summarily neglected EXCEPT for the case of purely imaginary eigenvalues. You should know why this special case needs special attention.
- Limit cycles! What is a limit cycle? What are tell tale signs that there is a limit cycle lurking in your two-dimensional phase plane sketch?

3. **Chapter 6, Laplace Transform**. You will be given a xeroxed copy of the table on page 300.

   - You must know the definition of Laplace transform. Even though the table on page 300 is given to you, I will ask you to work out some of them long hand.
   - You need to refresh your partial fractions. You will need it.
   - You need the *convolution integral* which is item #16 on the table.
   - You need to know how to use the step function $u_c(t)$ to extract parts of a given function $f(t)$.
   - You need to know the Dirac Delta function $\delta(t - t_o)$ and how to compute its Laplace transform.
   - You must know how to evaluate the following integral

     $$\int_0^t f(\tau)\delta(\tau - 17.85)d\tau \tag{5}$$

     for $t < 17.85$ and for $t > 17.85$.

4. **Chapter 5. Series Solutions of Second Order Equations**. The inspired guess is: the solution near the point $x = x_o$ is a power series of $(x - x_o)$! The attention here is limited to linear, second order ODE's:

   $$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x). \tag{6}$$

   Without loss of generality, we set $x_o = 0$.

- If $P(0) \neq 0$, then $x = 0$ is an ordinary point. The power series is a Taylor series. The only trick you need to practice is how to shift the index of summation.
- If $P(0) = 0$, then $x = 0$ is a singular point. There are two kinds of singular points: regular and irregular.
- You need to know how to identify a regular singular point (use the definition given in the book). Once it is identified as a regular singular point, an inspired guess for the solution is the product of x^r and an innocent ordinary-looking Taylor series. Here r is called the exponent of the regular singular point. It is determined by the *indicial equation* which is a quadratic (for second order ODE's).
- The mechanics of working such problems is straight forward: use the trick of shifting the indices of summation to get "all" the terms under a single summation sign with the same x^n factor, any "left over" terms are then written out explicitly. Setting each of the leftover terms to zero gives you the "indicial equation," and explicit values of some early coefficient. Setting the coefficient of x^n in your big summation term gives you the "recurrence relation." Viola, its done.
- If $r_1 > r_2$ and $r_1 - r_2$ is not an integer, then you are in luck. You need to be able to do such problems, getting two linearly independent solutions using $r = r_1$ and $r = r_2$ one after the other.
- If $r_1 = r_2$ or $r_1 - r_2$ is an integer (both r_1 and r_2 are real), you need to know there is a problem, and you need to know where to go in Boyce and Diprima to find the inspired guess for the second solution.
- We did not cover irregular singular point. If you are confronted with such a singular point, you are more or less on your own to get your inspired guess. A good inspired guess is: try an exponential factor in place of x^r and see if it works.

Good luck!