MAE 305
Engineering Mathematics I
Princeton University

Assignment # 10

December 1, 1997

Due on Friday, 2PM, December 5, 1997

1. **Chapter 10, Partial Differential Equations and Fourier Series**. While this is a course on ODE's, this chapter will show us that what we have learned, with appropriate modifications (*i.e.* we shall deal with two-point boundary-value problems) instead of *initial-value problems*), can be used to solve *partial differential equations* (PDE). Most importantly, we will learn how to use Fourier Series (and lots of other series) to represent an "arbitrary" *piecewise continuous* function in a finite interval. You will meet the concept of *eigen-value* again; instead of *eigen-vectors*, here you will meet *eigen-functions*.

2. **Chapter 10.1. Separation of variables; Heat Conduction; pp. 543-550**: Let u represent temperature of a rod located in the interval $0 < x < L$. The one-dimensional unsteady heat conduction equation is (this is rather easily derived; ask me in class if interested):

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

where α^2 is always positive. You know the *initial temperature distribution*:

$$u(x, t = 0) = f(x), \qquad 0 < x < L. \tag{2}$$

You also know what is happening at the two ends ("boundary points" are $x = 0$ and $x = L$. See eq.(4) on page 545). The mathematical question is: how to find $u(x, t)$ in the rod for $t \geq 0$? There are many ways to solve this problem, but we shall deal with only one of the simplest methods (and probably the most popular one): the *method of separation of variables.*

- Problem 5 on page 551. Just follow the procedures used in Example 1 on page 549. You will agree with me that this is a (triple) mickey mouse problem.

3. §10.2. Fourier Series. pp. 552-560. By now, you know that if the initial (temperature) distribution is given as summation of "special" sines and cosines (like the previous homework problem), then the solution is easy. This section tells you how to represent any arbitrary piecewise-continuous $f(x)$ into the summation of "special' sines and cosines. You must look at eqs.(6,7,8). You can readily verify that these relations are true. Look at the definition of *orthogonality* of $u(x)$ and $v(x)$ in the interval $\alpha \leq x \leq \beta$ as given by eq.(5) on page 554. The Euler-Fourier formula, eqs.(13) and (14), is easily derivable once you know eqs.(6,7,8).

   - Do problems #14 and #15 on page 561. These are straightforward.

4. §10.3. The Fourier Convergence Theorem. pp. 563-567. Read Theorem 10.3.1 on page 564—which gives the sufficient condition for the convergence of a Fourier Series—after studying the definition of *piecewise continuous* near the bottom of page 563. This theorem is very much needed because the usual "ratio test" of convergence may be "inconclusive." No proof of this theorem is given in B&D (but it is not that hard or abstract; ask me for a sketch of the "proof" if you are interested). Note that both $f(x)$ and $f'(x)$ are required to be piecewise-continuous. Note that each term of a Fourier Series is a perfectly continuous function of x, but the converged sum of the infinite series may be discontinuous! Read the section just above Fig. 10.3.3 on the Gibb's phenomenon (which says that in the neighborhood of a discontinuity, the value of the "partial sum" is wiggly).

   - Do problem #15 on page 568. You may peek in the back of the book and get the Fourier Series of your $f(x)$ from the answer of problem #1.

5. §**10.4. Even and Odd Functions. pp. 569-575.** Given a function $f(x)$ in a finite interval, we now know how to find its Fourier Series.

In general, it has both Sine and Cosine terms. What if I prefer, for personal private idiosyncratic reasons, Sines (or Cosines) only? Yes, you may accommodate your preference (by an honorable trick)! (see page 573 just below Fig. 10.4.3).

- Problem #7 and #9 on page 575. Note: you are asked only to sketch the even and odd "extension" of f of period $2L$. Just the sketch.

6. §10.5 Other Heat Conduction Problems. pp.578-587. Now we are ready to tackle heat conduction problems with arbitrary initial temperature distributions. All we need to do is to represent the initial temperature distribution $f(x)$ into a Fourier Series!

   There are two minor tricks here. The first deals with "boundary values" of u at the boundary points $x = 0$ and $x = L$. Previously, we had assumed $u(0) = 0$ and $u(L) = 0$. What if the boundary values are not zeros? The section on *Nonhomogeneous Boundary Conditions* pp.581-583 shows you what to do. What if their slopes are zero there instead? The section on *Bar with Insulated Ends* (pp.583-586) shows you what to do. What about more general problems? See *more general problems* on pp.586-587.

   - Problem #4 on page 587. Note: both ends of the bar are maintained at zero degree centigrade at all times. Thus, the initial temperature distribution given does not include the two end points. Do you agree that a sine series is the intelligent and correct choice?

7. §10.6 The Wave Equation. pp. 590-587. Given a violin string of length L and which is supported at its ends. At $t = 0$, you pluck it. The PDE that describes the displacement of the string from its "rest" position is

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{3}$$

   One of the many methods for solving this *wave equation* is the method of separation of variables. Fourier Series is again useful. You have all the tools and the procedures involved is entirely similar to the previous case.

- Problems #1 on page 597. You plucked the violin string at the center! You may omit (d) unless you have animation talent. Obviously, Matlab is useful for doing (b) and (c).

8. §10.7 Laplace Equation. pp. 602-609. The Laplace Equation in two dimension is given by eq.(1) on page 602. Mathematically, the triad of heat conduction, wave, and Laplace equations are also called *parabolic*, *hyperbolic*, and *elliptic* equations. I hope to show that the rationale for choosing this set of names in class. On page 603, you are told that if the values of u are prescribed on the boundary, the problem is called a *Dirichlet Problem*. If the values of the normal derivatives are prescribed, the problem is called a *Neumann Problem*. Just to make sure you know the names of the great mathematicians in this field. Again, one of the many methods available to solve the Laplace Equation is separation of variables. Again the Fourier Series is useful. You have all the tools.

   What happens if your region of interest is not a rectangle, but a circle? You need to find the Laplace equation in polar coordinates (see eq.(18) on page 607). How does one go about finding these equations in "other" coordinate systems? This is a substantial topic which we shall not have time to cover. You will most likely encounter this in your engineering courses. You will discover that Fourier Series is not the only game in town (to represent a function).

   - Problems #1 on page 609. Just do (a) and (b).