MAE 305
Engineering Mathematics I
Princeton University

Assignment # 11

December 5, 1997

Due on Tuesday, 2PM, January 6, 1998; After the Xmas Break

1. **This is the last week of lectures!**

   I am tentatively scheduling one review session on January 7th, 1998, Wednesday, at 10AM in the PMI Auditorium. However, watch the course Web page for any last minute changes of time and place.

2. **Chapter 11, Boundary Value Problems and Sturm-Liouville Theory**. This chapter generalizes a lots of stuffs that we have previously learned and accepted in Chapter 10.

   This is a list of the issues being explored:

   We learned that Fourier Series can be used to represent any piecewise continuous function $f(x)$ over a finite interval $0 \leq x \leq L$. Can "other kinds of series" do the same thing? The answer from this chapter is yes.

   We learned the word "orthogonal" on page 554 in describing eqs.(6,7,8) there: We say $\sin(n\pi x/L)$ is "orthogonal" to $\sin(m\pi x/L)$ when $n \neq m$, etc. (m, n are integers). We knew eqs.(6,7,8) are true because we can do the indicated integrations explicitly and prove to ourselves that it is true. Can we prove it for the functions used in "other kinds of series?"

   Then there are new (and important) stuffs:

   - $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ are "eigenfunctions" of the ODE:

   $$\frac{d^2y}{dx^2} + \lambda_n y = 0.$$

where $\lambda_n = (n\pi/L)^2$ is the n-th eigenvalue, and n is any integer. It turns out that this is a special case of the Sturm-Liouville Equations.

- The concepts of *adjoint operator*, and *self-adjoint operator*. We will see that the Sturm-Liouville equation is simply the most general form of a second order linear differential self-adjoint operator.
- In this chapter, we shall show that the eigenfunctions of a Sturm-liouville equation can be used to construct an infinite series to represent any piecewise continuous function in an interval.

So here we go.

3. **Chapter 11.1. The Occurrence of Two-point boundary value problems; pp. 621-624**: Light reading

    - Problems #8, #9 on page 624.

4. §11.2. Linear Homogeneous Boundary Value Problems: Eigenvalues and Eigenfunctions. pp. 625-630.

    - Do problems #15 on page 631. You may not know it, but what you are doing here is a generic procedure we will do it many more times later in abstract form.

5. §11.3. Sturm-Liouville Boundary Value Problems. pp. 633-643. I want to supplement the book and introduce to you the concept of an *adjoint* operator ($\mathcal{L}^*$) to a linear second-order differential operator ($\mathcal{L}$).

    We follow the books notation. The interval of interest is $0 < x < 1$. Let $u(x)$ and $v(x)$ be real functions of x in this interval. The *inner product* of u and v (an integral over the interval of interest involving the product of $u(x)$ and the complex conjugate of $v(x)$) is denoted by (u, v) and is defined by eq.(7) on page 235. We now write down a generalization of the Lagrange's identity (eq.(5) on page 634) as follows:

    $$(\mathcal{L}(u), v) - (u, \mathcal{L}^*(v)) = [\,\textit{something}\,]_{x=0}^{x=1} \tag{1}$$

    What is $\mathcal{L}^*$ and what is the bracketed "something" on the right hand side? Well, since $\mathcal{L}$ is a second order linear differential operator, the first term on the left hand side can be integrated using integration by parts a few times until no derivative of u appears anymore under the integral sign. All the stuff you integrated out is put on the right hand side. All the stuff you left under the integral side (involving $v(x)$ and its derivatives) is represented by the second term on the left hand side. The above instruction will give you both the *adjoint operator* $\mathcal{L}^*$ and the exact formulae for the "something" on the right hand side.

If you now require both u and v to satisfy "homogeneous boundary conditions," the "something term" disappears from the right hand side (study the bottom of page 634!). We then have:

$$(\mathcal{L}(u), v) = (u, \mathcal{L}^*(v)). \tag{2}$$

If you discover that the $\mathcal{L}^*$ you just found by going through this exercise is identical to $\mathcal{L}$, you can conclude that $\mathcal{L}$ is a *self-adjoint* operator.

Example: if $\mathcal{L}(u) = \frac{d^2u}{dx^2} + \frac{du}{dx}$, then $\mathcal{L}^*(v) = \frac{d^2u}{dx^2} - \frac{du}{dx}$. Hence this $\mathcal{L}$ is Not self-adjoint.

Well, the Sturm-Liouville linear second order differential operator is a self-adjoint operator. The eigenfunctions of a self-adjoint operator are orthogonal to each other, ... (remember $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$?). That's why it is special. Note that $p(x)$ and $r(x)$ are assumed *not* to change sign in the interval of interest.

- Do problem #14 on page 643. I want you to display the adjoint operator $\mathcal{L}^*$ that you derived.

6. §**11.4. Nonhomogeneous Boundary Value Problems. pp. 646-655.** You want to solve eq,(1) on page 646. The $f(x)$ on the right hand side is a mess. What do you do? You solve eq.(3) as a eigen-problem: find the eigenvalues and the eigenfunctions. First, you represent the messy $f(x)$ by a series of eigenfunctions. Then apply the method of undetermined coefficients by using your inspired guess of an infinite series of eigenfunctions for your inhomogeneous solution! (See how the Fourier Series case fit into this grand scheme of things?).

   - Problem #1 on page 655. As I look through the problems here, the eigenfunctions are all sines and cosines. This is probably because its too much work at this point to ask you to find your very own set of eigenfunctions for some special self-adjoint $\mathcal{L}$. Note that this problem can be solved very easily using other techniques. Here I want you to go through the eigen-analysis.

7. §11.5 Singular Sturm-Liouville Problems. pp.661-667. Light reading. This section essentially provides justification for the study of series solution near singular points in Chapter 5.

8. §11.6 Further Remarks on the Method of Separation of Variables: A Bessel Series Expansion. pp. 669-672. Just to show you that Bessel Function is useful stuff in polar coordinates. Its the Fourier Series of polar coordinate problems.

    - Problems #4 on page 673. The left hand side of the given equation is the two-dimensional Laplacian, $u_{xx}+u_{yy}$. We have not discussed how to rewrite the Laplacian in "other" coordinates. If you are interested in how to do this, ask me in class.

9. §11.7 Series of Orthogonal Functions. pp. 675-680. The neat result here is eq.(9) on page 677. Applying it to the Fourier Series, we see that it is precisely how we determine the coefficients of a Fourier Series. But deriving it in this elegant way gives a finite sum series special meaning: if for whatever reason you decided to take only N terms to represent a function $f(x)$, evaluating the coefficients by eq.(9) also says your result has minimum mean-square error (in comparison to any other alternative evaluations) for a N term result.

    - Problems #1 on page 680. Note: $S_n(x)$ is the sum of n terms of some sequence. Here, you see an interesting example of the subtleness of limiting processes. If you take the limit of $n \to \infty$ first before integrating, you get one answer. If you integrate first then take the $n \to \infty$ limit, you get another answer. Whenever you are evaluating a term involving with a limiting process, the limiting process is always done last. If you have more than one limiting processes, the answer in general can depend on the order the limiting processes are executed.