

MAE 305
Engineering Mathematics I
Princeton University

Assignment # 3

September 26, 1997

Due on Friday, 2PM, October 3, 1997

1. **Where we are.** We have learned how to deal with a single first order ODE for a single dependent variable in the past two weeks. There is a short list of four such ODEs that you know how to deal with analytically: **linear** (use integration factor), **separable** (separate them across the equal sign), **exact** (make sure it passed the “test” that μ does NOT depend on BOTH the dependent and independent variables—see page 87), and **homogeneous** (if it passed the “scaling test”—if both x and y are changed by the same scaling factor, the ODE does not change. You introduce a new variable $v \equiv y/x$, and the ODE for v is separable). Then you know how to recognize and deal with the nonlinear **Bernoulli’s equation** (work with a new dependent variable for which the ODE is linear), and you know the subtleties of the wonderful **Clairnaut equation** (why is this equation wonderful?). You also know the *Lipschitz Condition* ($|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$; M is a finite positive constant), and the marvelous concept of “iteration,” or “sneaking up on a solution” via a limiting process on a *sequence* of iterants. If the right hand side of a quasi-linear first order ODE satisfies the Lipschitz condition in a box which includes the initial point, a solution satisfying the initial condition and the given ODE *exists* and is *unique*. In addition, for any first order quasi-linear ODE, you are now competent to obtain a numerical solution using some software package, and plot the solution.
2. **Numerical Methods.** You know **Runga-Kutta** is a most popular algorithm (both ode23 and ode45 are variants of R-K). It is the method of choice when you don’t have a “stiff problem.” For stiff problem,

“backward difference” is the key. A popular algorithm is called “Gear” (The Director of Research at NEC Research Lab on Route 1). Mathematica has Gear (when you use “NDSolve,” Mathematica picks the best method automatically). The last time I checked, Matlab 4.2c does not have Gear.

3. **Readings.** Boyce and DiPrima, Chapter 3: Second Order Linear Equations. Basically, this is very simple stuff. In essence, we are dealing with a one dependent variable (y) second order linear ODE.

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

This problem is identical to the following two dependent variables (y and v) ODE problem:

$$\begin{aligned}\frac{dy}{dt} &= v, \\ \frac{dv}{dt} &= g(t) - (p(t)v + q(t)y).\end{aligned}$$

I expect you to say to yourself right now: “Aha, I can do this problem on the computer!” And indeed you can. In fact, you can do any second order quasi-linear ODE on the computer (the right hand side of the second equation can be as complicated as you wish)—provided it is posed as an “initial value problem.” (What happens if you want your solution to honor $y(0) = A$ and $y(5) = B$? You need Chapter 11!)

Because of second order linear ODE is so frequently encountered in practice, we shall devote a whole week to its analytical properties. We will work a little bit on “how to” find solutions. The fun part is to show “what” a second order linear ODE is capable of doing.

4. **Comments on Readings.**

§3.1: Homogeneous Equation with Constant Coefficients. Read pp. 121-128. **Warning!** The word “homogeneous” here means something completely different than the same word used in Chapter 2! (Here it means $y = 0$ is a solution to the equation). Here, it means $g(t)$ is absent! In fact, $g(t)$ is called the “nonhomogeneous

term” in Boyce and Diprima. I sometimes call it the “forcing term.” You learn here that, when p and q are constants, the solution is (nearly always) in the following form:

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where r_1 and r_2 depend on p and q , and C_1 and C_2 are integration constants ($r_1 \neq r_2$ assumed). Note that r_1 and r_2 may be complex (even though we insist that the solution is real!).

§3.2 Fundamental Solutions of Linear Homogeneous Solutions.

pp.130-139. The Principle of Superposition is a big deal. Given a linear homogeneous equation $L(y) = 0$, and two “linearly independent” solutions $y_1(t)$ and $y_2(t)$. The punch line is: $y_3(t) \equiv C_1 y_1(t) + C_2 y_2(t)$ is also a solution!! If $L(y) = 0$ is a second order ODE (I didn’t say it was until now), then y_3 is the *general solution*! Wow! For the skeptics: check it out! To fully understand what was said above, you need to understand the concept of “linear independence,” the concept of general solution, and a little bit on the concept of “Wronskian.”

§3.3 Linear Dependence and the Wronskian. Given two solutions of a second order ODE. How do I know they are linearly independent? Give it the Wronskian test! Remember: if the Wronskian is ever zero somewhere, it is zero everywhere (Theorem 3.3.3, which follows from 3.3.2. Abel was a great guy).

§3.4, Complex Roots. pp.145-149. I presume you already know:

$$\begin{aligned} \cos(t) &= \frac{e^{it} + e^{-it}}{2}, \\ \sin(t) &= \frac{e^{it} - e^{-it}}{2i}, \end{aligned}$$

and

$$\begin{aligned} e^{it} &= \cos(t) + i \sin(t), \\ e^{-it} &= \cos(t) - i \sin(t), \end{aligned}$$

where i is the square root of minus one. This section just shows you what an exponential function with a complex exponent looks

like. Do you know what “hyperbolic functions” are? Here is a reminder: $\cosh(t) \equiv (e^t + e^{-t})/2$ and $\sinh(t) \equiv (e^t - e^{-t})/2$. What second order ODE does each one satisfy?

§3.5 Repeated Roots pp.153-159. What happens when $r_1 = r_2$? The solution is no longer the sum of simple exponentials. Look at equation (11) on page 154. You can skip from here to page 158. The good stuff is **Reduction of Order**: If you know one solution somehow, you can get the second one cheap!

§3.6-3.7, Nonhomogeneous Equations. pp.163-170 and pp.172-176. What to do if $g(t)$ is not zero? Here comes two methods of handling it. The first method (undetermined coefficients) needs an inspirational guess. The second method (variation of parameters—due to the great Lagrange) requires that you know the homogeneous solution. It is generically the same as the “order reduction” method under Repeated Roots.

The Rest of the Chapter. There are tons of examples. Scan then at your leisure—knowing that you can do any of them if you are asked to! Take a special look at Figure 3.9.3 and the materials associated with it. The magic word is: **resonance!**

5. **Home Work Problems.** On some of the problems, I am requiring you to also get me a numerical solution or plot the analytical solution (just to keep you in touch with the computer). Learn to dress up the plots by reading the Matlab User’s Guide in the EQ Library—it’s useful for all your other courses.

Page 128. Homogeneous Equations with Constant Coefficients.
Pretty straightforward stuff.

- Problems (19): Are $\sinh(t)$ or $\cosh(t)$ solution to this ODE? Use Matlab or some computer software to plot the graph. Dress it up a bit.
- Problem (30): You got to read the paragraph just above to learn this cheap trick (#1).
- Problem (34): You got to read the paragraph just above to learn another cheap trick (#2).

So, start another short list! You now have two special second order nonlinear ODE's that you can handle!

Page 138. Fundamental Solutions. Mostly Wronskians.

- Problem 24: Straightforward.
- Problem 28: You got to read the paragraph just above to learn another cheap trick (#3)—this one is for linear equations only.

Just note the word “Adjoint” (equation) in problem 32. We will meet it again.

Page 144. Linear Independence. Straightforward.

- Problem 17. Students, meet Bessel's equation. The point here is: you can find the Wronskian without knowing the two solutions the Wronskian is made out of! Use equation (12) on page 142—after reading the materials.

Page 150. Complex Roots. Add one more cheap trick to your short list.

- Problem 24: Do it analytically, and also do it numerically using Matlab or Mathematica. Plot it, of course.
- Problem 39: You got to read the paragraph just above to learn this cheap (Euler) trick (#4).

Page 159. Repeated Roots. Use what you learn on page 158.

- Problem 23: Straightforward.

Page 177. Variation of Parameters. Undetermined Coefficients included here.

- Problem 1: Straightforward.

You are expected to do all 10 problems. You have a short list of 4 cheap tricks, not counting the method of undetermined coefficients (needs inspiration) and variation of parameters (needs the homogeneous solutions).

6. We hope to spend some time with §3.9 **Forced Vibrations**. What happens to the dependent variables of Equation (1) on page 193 when ω , the frequency of the nonhomogeneous term, changes? Look at fig. 3.9.1, 3.9.2, and 3.9.3!