

MAE 305
Engineering Mathematics I
Princeton University

Assignment # 4

October 3, 1997

Due on Friday, 2PM, October 10, 1997

1. **Erratum.** In assignment # 3, under **Comments on Readings . . . §3.4**, there is a typo. The equation relating $\sin(t)$ to the exponentials should be:

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}. \quad (1)$$

2. **Where we are.** For linear second order ODE's, we learned that the solution can be expressed as the sum of the homogeneous solution and the nonhomogeneous solution. Since the ODE is second order, the general solution contains two integration constants. The homogeneous solution for a second order linear ODE can be expressed as

$$y_{homog} = C_1 y_1(x) + C_2 y_2(x) \quad (2)$$

where C_1 and C_2 are the two integration constants (they are used to satisfy imposed initial conditions), and $y_1(x)$ and $y_2(x)$ are *linearly independent* solutions of the homogeneous equation. How do we know two given functions of x are linearly independent? Evaluate their Wronskian at any one single point!

You have learned several methods to solve for $y_1(x)$ and $y_2(x)$ analytically, particularly when all coefficients are constants.

You have learned two methods to get the nonhomogeneous solution. The method of undetermined coefficients (you need an inspirational guess of the terms needed), or the method of variation of parameters (you need at least one but preferably both of the linearly independent homogeneous solutions).

We have also learned four (cheap) tricks for four special kinds of second order ODE's: (a) If the dependent variable does not appear, you can reduce the order from two to one. Whether this helps or not depends on whether you know what to do with the resulting first order ODE. This trick does not need the problem to be linear. (b) If the independent variable does not appear, you can reduce the order. Same comments as (a) apply here. (c) For a linear problem, if the given $P(x)$, $Q(x)$ and $R(x)$ satisfy $P''(x) - Q'(x) + R(x) = 0$ (see page 138-139), then you are in luck. (d) If the coefficient of the k -th order derivative is x^k (where x is the independent variable), then you have *Euler's equation* and you can use the Euler trick (changing the independent variable). You are expected to know cases (a), (b) and (d) by heart, but you do not need to memorize the details of (c); I just want you to know trick (c) exists, and can find it from Boyce and DiPrima if you ever need it (If you lost or sold your Boyce and DiPrima after this semester, it should not be difficult for you to rederive the trick).

3. **The Constant Coefficient Case.** We examined the case of constant coefficient second order linear ODE with a sinusoidal nonhomogeneous $g(t)$. We learned to interpret the coefficient of the first derivative as the damping "constant," and the coefficient of the second derivative as the spring "constant" (assuming unity mass for the coefficient of the second derivative). The meaning of the sign and magnitude (and the physical dimension) of these two numbers are understood. The concepts of "natural frequency" and "damping" are now understood. The homogeneous solution will blow up if the spring constant is negative, or/and if the damping coefficient is negative. Such problems are "unstable." The response of the dependent variable to a stable problem (both coefficients are positive) to a sinusoidal nonhomogeneous term with frequency ω have been studied and understood. The words "resonance" and "phase shift" should have some meanings to you (otherwise, ask me).
4. **New Readings.** Boyce and DiPrima, Chapter 4: Higher Order Linear Equations. Again, this is very simple stuff. In essence, we are dealing

with a one dependent variable (y) linear ODE of the following form:

$$\frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + p_n(t) y = g(t)$$

where $n \geq 2$. This problem is identical to the following n dependent variables (y_1, \dots, y_n) system of ODE problem:

$$\begin{aligned} \frac{dy_1}{dt} &= y_2, \\ \frac{dy_2}{dt} &= y_3, \\ &\vdots \\ \frac{dy_n}{dt} &= g(t) - (p_1(t)y_n + \dots + p_n(t)y). \end{aligned}$$

I expect you to say to yourself right now: “Aha, I can do this problem on the computer for any reasonable n ” And indeed you can—provided it is posed as an “initial value problem” for all of the dependent variables. The detailed and more advanced theories (which provide “understandings”) of a system of linear first order ODE’s is to be covered separately in Chapter 7, and not in this chapter. We will do Chapter 7 next week. For this week, we just generalize what we learned in Chapter 3.

5. Comments on Readings.

§4.1: General Theory of n -th Order Linear Equations. pp.203-206. Nothing tricky here at all. The representation of the solution as the sum of homogeneous solution (with n integration constants) and a single nonhomogeneous solution is easily generalized. The Wronskian’s rule in the litmus test for linear independence is easily generalized.

§4.2 Homogeneous Equations with Constant Coefficients. pp. 208-214. Aha! $y = e^{rt}$ is again an inspirational guess for the homogeneous solution when all coefficients are constants! Now we have a n -th order *characteristic polynomial* for r . Setting it to zero (why?) yields an algebra equation for r , yield n solutions, commonly known as *roots*. You learn what happens if the roots

are real and distinct (pretty much the same as the second order case), and when they are complex (you need to be able to convert between sine, cosine and exponentials with imaginary exponents). And you learn what happens if there are repeated roots (instead of “pure” exponentials, we now have exponential times a polynomial). Look at Example 4 on page 213 to see how to take the fourth root of -1 . After studying this, can you find the fifth root of $2 + i$? If not, ask me in class.

§4.3: The Method of Undetermined Coefficients and **§4.4: The Method of Variation of Parameters.** pp.216-219; 221-222. Completely straightforward generalization.

6. **Home Work Problems.** You are encouraged to consult with classmates and work together. But all work submitted must be done by you—after talking things over. You are on your honor that you do not submit copied work.

Page 200: End of Chapter 3. This can be fun!

- Problem (25). Here comes the Duffing Equation! You have a positive damping coefficient $\mu = 1/5$, a positive linear spring constant $k = 1$ (thus the natural frequency ω_o is unity), and a nonlinear term ϵu^3 where $\epsilon = 1/5$. You can interpret the last two terms on the left hand side with $\epsilon > 0$ as the force of a nonlinear spring which “hardens” or becomes stiffer when deformed (What happens to the nonlinear spring if ϵ is negative?). There is no known analytical trick to solve the nonhomogeneous problem exactly. So we go to the computer. Part (a) asks for the *steady response*. The solution driven by a *periodic* nonhomogeneous term is said to have reached its steady response when it appears to repeat itself in some periodic fashion. Does it always happen? If so, how long do you have to wait until it happens? Does the (periodic) solution depend on the initial condition? (Aha! what do you think?). Since we have no analytical answers, we have to use the computer to generate the solutions. Hopefully we can get some answers to the above questions by playing “what if” games. Does the solution become “steady” after 10 cycles?

50 cycles? 2000 cycles? Does the amplitude of the steady response depend on the initial condition? How confident are you about whatever conclusions you may reach by looking at your computer runs?

Do what the book asked you to do. But if you should find it fun to mess around, then mess around. The Duffing equation is a great equation to mess around.

Pages 206-208: General Theory . . . Here is a generalization of the “reduction of order” method if you (somehow) know ONE homogeneous solution.

- Problems (26): What happened to the term involving v itself? The coefficient is the left hand side of the homogeneous equation operating on y_1 , and therefore vanishes. You just need to do it once yourself to remember what happens.
- Problem (27): This is for practice. How did one guess the y_1 ? There is no substitute for inspiration!

Pages 214-216: Homogeneous Equations with Constant Coefficients. Quite straightforward.

- Problem 17: A problem with repeated roots. Now what happens when there is repeated roots?
- Problem 29: How many integration constant do you expect in a general solution, and how many have you got here? How do you prove a solution is a solution?
- Problem 39: The problem itself took half a page. But don't be intimidated. Just do as you are told for part (a). Get the general solution for part (b). Now that you have the general solution, you need to determine the integration constants. That's part (c) and part (d).

Pages 219-221: Undetermined Coefficients. Straightforward.

- Problem 10. You need an inspirational guess for the “form” of the nonhomogeneous solution $Y(t)$. This problem does not need a genius to get the right inspiration. The homogeneous solution for this problem is a simple one.

For the adventurous: bottom of page 219 to top of page 221 is the method of “annihilators.” It really needs a spectacularly in-

spirational guess in order to use it. The class is not required to study this. Just be informed that an “annihilator” is not a “power ranger” toy.

Page 224: Variation of Parameters. A straightforward generalization of the method first learned in the second order case. Just a bit more messy.

- Problem 2: I picked the easiest one here. Once you do this, you can do any of the others. They are all the same.

There are only 8 problems assigned. The first one on the Duffing equation asks you to use computer generated solutions to find out “what happens.” The rest of the seven is to exercise analytical “how to” methods. Have fun.