MAE 305
Engineering Mathematics I
Princeton University

Assignment # 7

October 31, 1997

Due on Friday, 2PM, November 7, 1997

1. **Chapter 9**, Nonlinear Differential Equations and Stability]: This is fun stuff. Instead of focusing on "how to" find exact solutions, the emphasis here is on "what to expect" from the solutions qualitatively.

2. §9.1. **The Phase Plane: Linear Systems. pp. 459-468.** You will notice that the discussion here is limited to two dependent variables, x_1 and x_2. A direction field plot of x_2 versus x_1 is now called a *phase portrait* drawn on the *phase plane*. This the is language agreed upon by all the world's scholars in this field (it is actually a terminology originated in physics). What happens when you are dealing with more than two dependent variables? The answer is: the same general ideas apply, but things are much more complicated! So, do not casually extend the conclusions here lightly to more than two dimensions. When the two dependent variable problem is linear, constant coefficient and homogeneous, you learn here: (a) How to associate the graphical picture of the phase portrait with the eigenvalues r_1 and r_2 of the matrix $\mathbf{A}$. A summary of the whole thing is given by Table 9.1.1 on 468. (b) The precise definition of stable, unstable, and asymptotically stable singular points will be clarified in the next section. (c) It is easy to show why linear homogeneous systems are building blocks to understand much more complicated nonlinear two dependent variables problems.

   - Do problems #3, #6, #9 and #14 on page 469. In #14, you are given a nonhomogeneous equation. You need to "shift the origin" and work with a new dependent variable vector $\mathbf{u}$ with components u_1 and u_2.

If you use Matlab to plot the phase portrait, remember that you once learned how to "hold on" a graph, and to plot additional stuff on the same graph. Use

$$plot(t, y(:,1), t, y(:,2))$$

to plot y(1) versus y(2).

3. §9.2. **Autonomous System and Stability. pp.471-478.** Given a system of quasi-linear ODEs for the vector $\mathbf{x}$, the system is called *autonomous* if the right hand side depends only on $\mathbf{x}$ and does not depend on t.

   I will show in class that any nonautonomous system can be transformed into an autonomous system (at the cost of increasing the "dimension" of $\mathbf{x}$. If I forgot, ask me. The whole mission of this section is to clarify the meaning of stability and instability. The obvious intuitive ideas of stability and instability are OK—until the idea of *asymptotically stable* critical point shows up. A critical point is asymptotically stable when a trajectory starting from a region enclosing the critical point eventually arrives at the critical point itself as time goes to infinity. But before the point goes there, however, it may have gone anywhere. A simple stable critical point is when such trajectories keeps a finite "distance" from the critical point as time goes to infinity.

   - Problem #7, page 478. There are four critical points here!

4. §9.3 **Almost Linear Systems. pp.480-488.** What happens if your problem is not really linear, and the linear system is only an approximation in the neighborhood of a critical point? The answer is given by Table 9.3.1 on page 484. The punch line is: everything you found out from the linear system is applicable to the almost linear system *except when all eigenvalues are precisely imaginary.* See the bottom line of the table.

   - Problem #12, page 488.

5. §9.4 **Competing Species. pp. 493-501.** Competing species is just an excuse for studying eq.(2) on page 493. We will study Example 2 on page 496—to learn about this thing called the *separatrix*. Look at Fig. 9.4.4 on page 499. By studying the phase portrait, we figured

out that the final home of the trajectory depends on which side of the separatrix the initial condition was on!

6. §9.5 **Predator-Prey Equations. pp. 504-510.** We will skip this section. It studies the same general system of ODEs with different numerical values for the constants. You may read it for fun.

7. §9.6 **Liapunov's Second Method. pp. 512-520.** We will skip this (important) section for two reasons. First of all, it will be introduced in application courses (Control theory courses) where they are used extensively. Secondly, the method requires again an "inspired guess" of what the Liapunov function V is, and all readily concedes that there is no methodical way of finding the Liapunov function (see Boyce and Diprima, page 516, second line from the top.

8. §9.7. **Periodic Solutions and Limit Cycles. pp. 522-527.** Now, this is important stuff! What happens if a critical point looks unstable when you look at it with a microscope, but looks stable when you look at it with a telescope? Answer: LIMIT CYCLES! Look at Example #1. What happens to eq.(4) on page 523 when $x^2 + y^2$ is very small (looking with microscope), and when x^2+y^2 is very large (looking with telescope)? Figure 9.7.1 is the answer.

    - Problem 8 on page 531. An obvious hint is: get an equation for $r = \sqrt{x^2 + y^2}$. In general, it is not always so easy to change a limit cycle problem in two dimension into a single autonomous ODE in terms of the radius. So the simplicity of this section is somewhat misleading.

9. §9.8 **Chaos and Strange Attractors: The Lorenz Equations. pp. 533-539.** In the two dimensional autonomous world, there is *no chaos*—regardless of how nonlinear the right hand side is. If you want chaos, you need at least three dimensions—and a nonlinear right hand side. Read this section for fun. I will talk a bit on it.