MAE 305
Engineering Mathematics I
Princeton University

Assignment # 8

November 7, 1997

Due on Friday, 2PM, November 14, 1997

Please come in to see me if you did not do well in the first mid-term.

1. **Chapter 6. Laplace Transform**: After a taste of nonlinear problems, we are back onto some straightforward stuff. This chapter talks about *constant coefficient* linear ODEs with an nonhomogeneous term for one single dependent variables. Aha! You would say: we know how to do such problems! Get the eigen stuff, diagonalize, and use variation of parameters or method of undetermined coefficients. Why spend another chapter on this stuff?

2. §6.1. Definition of Laplace Transform. pp. 289-295. The answer is: **Laplace Transform**. It is yet another way of solving this class of problems, and many, many branch of engineering use this methods extensively. Once you learn this method, you may like it so much that you would wonder why anyone would solve this class of problem in any other way.

   In order to use the method of Laplace Transform, we need to know what it is. In principle, you need a whole course on *complex variables* to do justice to Laplace Transform. But it is possible to do Laplace Transform before you have a full course on complex variables.

   This is the definition of Laplace Transform (see eq.(2) on page 290): Given $f(t)$, its Laplace Transform is denoted by $F(s)$ or $\mathcal{L}[f(t)]$ which is defined as

$$\mathcal{L}[f(t)] = F(s) \equiv \int_0^\infty \exp(-st) f(t) dt. \tag{1}$$

In English, (1) says: Give me a $f(t)$ (defined for $t > 0$), and I will give you a $F(s)$ called its Laplace Transform (defined for $s > 0$)—provided the integral exists.

So you need to learn how to find $F(s)$ given a $f(t)$.

- Do problems #5, #7, #11 and #15 on page 294. These are straightforward. You need to know your integration by parts.

3. §6.2. Solution of Initial Value Problems. pp. 295-303. Of course, we will eventually need the "reverse" of (1), which we will express in the following form:

$$f(t) = \mathcal{L}^{-1}[F(s)]. \tag{2}$$

In English, (2) says: Give me a $F(s)$, and I will give you back your $f(t)$. Unlike (1) which told you explicitly how to get $F(s)$ (by doing the indicated integration), this (2) above does not tell you how to get it. Actually, the right hand side of (2), called the *inverse Laplace Transform*, is also defined in terms of a (complex) integral with $F(s)$ appearing in the integrand (and s being a dummy variable). But in MAE 305, we shall try to make do without involving the mathematical algorithm for doing the inverse Laplace Transform.

And here is the deal. Look at Table 6.2.1 on page 300. A Table! In fact, the answers of the homework you just did can be found there! How do you find $F(s)$ given a $f(t)$? You can look it up (knowing in your heart that you can find it yourself by doing the integration). How do you find $f(t)$ given a $F(s)$ in this course? You look it up (knowing that after you take MAE 306 you will be able to find it yourself by doing a *contour integral*. What is a contour integral? You need MAE 306!).

- Do problem #12 and #22 on page 303. These are straightforward—make sure you use Theorem 6.2.1 and 6.2.2. Just use the Table, and show off your partial fraction skills.

4. §**6.3. Step Functions. pp.306-310.** The *step function*, denoted by $u_c(t)$ in eq.(1) on page 306 (the step occurs at $t = c$), is a wonderful

function. What is its F(s)? It is given by eq.(2) on page 307. Of course, you can also find it in the Table on page 300 as item 12.

Now if you are given the *product* of $u_c(t)$ and $f(t-c)$ (remember, the step function "steps up" at $t = c$), what is its Laplace Transform? Of course, you can work this out (This is Theorem 6.3.1 on page 308). There are lots of subtleties here. But the bottomline is: the Laplace Transform of this particular product can be expressed by eq.(3), and its inverse transform can be expressed by eq.(4)—both on page 308.

- Problem #7, page 311. Use Theorem 6.3.1.

5. §6.4 Differential Equations with Discontinuous Forcing Functions. pp.313-317. Nothing fundamentally new here. Just more practice on Theorem 6.3.1

   - Problem #10, page 318. Use Theorem 6.3.1.

6. §6.5 Impulse Functions. pp. 319-323. What is the Laplace Transform of the *Dirac delta function*? The Dirac delta function, named in honor of Dirac, is a function of t which is zero everywhere except in the time interval $t_0 - \tau \leq t \leq t_0 + \tau$ where its value is $1/2\tau$—and τ is an asymptotically small number (*i.e.* it is smaller than the smallest nonzero number that just crossed your mind). It is denoted by $\delta(t-t_o)$, and its Laplace Transform is given by eq.(13) on page 322 (and as item 17 on the Table in page 300). Remember, the integral of the Dirac delta function over the narrow "pulse" is unity.

   - Problem #12, page 324. So it is a fourth order ODE. Some more practice for your partial fractions.

7. §6.6 The Convolution Integral. pp. 325-330. Question: if I give you the inverse transforms for $F(s)$ and $G(s)$ (*i.e.* $f(t)$ and $g(t)$), what can you tell me about the inverse Laplace Transform of $F(s)G(s)$? The answer is given by Theorem 6.6.1 on page 326. The proof on page 327 is pretty simple, even though it looks somewhat intimidating.

   You are also exposed to the notation of a *generalized product* defined by eqs.(2) and (3) on page 326. Some important properties are summarized by eqs.(3,4,5,6,7) on page 326. Remember, the generalized product "$*$" is just a short hand notation for writing eq.(2). You should not be surprised to find that $f * 1$ does not equal to f.

So why is the convolution integral stuff a big deal? If you need to find the inverse Laplace transform of some $K(s)$ that is not listed in the Table on page 300, the battle may not be lost. See if you can break $K(s)$ into a product of two factors each of which can be found from the Table, then use the convolution integral to give you the answer.

- Problems #9 and #10 on page 331. Totally straightforward.

8. So perhaps you can now see why Laplace Transform is a competitive method for solving constant coefficient linear ODEs. If we are always dealing with functions whose Laplace Transform and its inverse Laplace Transform can be found from a Table, we are ahead. If we are dealing with problems for which the Table is inadequate, we are dead (until we take MAE 306). It turns out that for many, many practical problems, the Table is adequate.

   Laplace Transform via the theory of complex variable is a *beautiful* subject. We glossed over much subtleties and other applications.