MAE 305
Engineering Mathematics I
Princeton University

Assignment # 9

November 14, 1997

Due on Friday, 2PM, November 21, 1997

1. **Second Mid-Term Next Next Week**! The second mid-term is scheduled on November 26th. It will cover Chapter 9 (nonlinear differential equations and stability), Chapter 6 (Laplace Transform), and Chapter 5 (Series Solutions). The classes on next Friday (11/21) and next-next monday (11/24) will be review sessions.

2. **Chapter 5. Series Solutions of Second Order Equations**: We are now experts on constant coefficient ODE's! In this chapter, we deal with non-constant coefficient second order (one dependent variable). The idea being explore is: how to go about representing the solution in terms of an infinite power series. Why should this be an idea worth exploring? There are two reasons. Firstly, you may be interested only in the solution only very near the point $x = x_0$, and a series solution is a cheap way to get a useful answer there. Secondly, there are points (called singular points) where simple-minded numerical methods will always fail. And you must use what you learn here to sort out what really happens there.

3. §5.1. Review of Power Series. pp. 225-230. I hope you remember your power series from your calculus courses. In any case, this section is a good review. Look at the ten items in §5.1.

   While all ten items are important, the following three are more important than the others. You must know the ratio test (#3), the concept of radius of convergence ρ (#5), and the definition of an *analytic* function (#10). Most of the rest basically tells you: if a power series representation of $f(x)$ and $g(x)$ are available, how to work with the two series term by term. The little "shift of index of summation" trick, so innocent looking on page 229, will be used extensively later.

- Do problems #9 and #15 on page 231. These are straightforward.

4. §5.2. Series Solution near an Ordinary Point, part 1. pp. 232-241. What is an ordinary point? Look at eq.(1) on page 232. Answer: if $P(x_o) \neq 0$, then x_o is an ordinary point. Lipschitz is happy. We can easily see that we have no existence and uniqueness problem with solutions in the neighborhood of an ordinary point. Now the whole idea of a series solution is just an exercise in "undetermined coefficients." The only new issue is: we now have an infinite number of undetermined coefficients! How do you find the infinite number of undetermined coefficients? The trick is: **recurrence relation**! (see page 234). The idea is: find one, and use what you have just found to find the "next" one, and use what you have just found to find the "next" one, etc. Look at Fig. 5.2.1,2,3,4. Series solutions are not useful when you are not near the point x_o because you will need to use a lot of terms to arrive at the "convergent" solution.

   - Do problem #21, (a) and (b) only, on page 242. What is new here is that, sometimes, when your lucky star is shining, the power series terminates! A finite power series is called a polynomial.

5. §**3.3. Series Solution near an Ordinary Point, part 2. pp. 243-247.** The mission of this section is generalize the definition of an ordinary point: we need $p(x)$ and $q(x)$ to be *analytic* at $x = x_o$. It is a subtle point to avoid some pathological situations.

   - Problem #10, (a) and (b) only, page 247. Do you know how to express
   $$\frac{1}{1 - x^2}$$
   in a power series about $x = 0$? Use long division (some people call it synthetic division)! Or, take a peek at example #1 on p.245.

6. §5.4 Regular Singular Points. pp.250-253. Now we deal with the case where $P(x_o)$ is zero. Lipschitz is unhappy. We need to be careful here.

   We shall deal with *regular singular point* here. (Irregular singular point is being dealt with only in problem 20 on page 274-275 of B&D). The definition of a regular singular point is given by eq.(6,7) (or, eq.(8)) on page 252.

- Problem #1 and problem #6, page 253. Totally straightforward.

7. §5.5 Euler Equations. pp. 255-261. You know Euler Equation, and it is seen to have a regular singular point at $x = 0$. The only "new" thing here is what to do if you have complex roots and repeated roots. A useful formula is

$$x = \exp(\ln x)$$

which is valid for complex x. For repeated roots, either undetermined coefficients (needs inspired guess, see eq.(10) on page 256) or variation of parameter will work. You should scan this section. We will not spend much time on it.

8. §5.6 Series Solutions near a Regular Singular Point, Part 1. pp. 262-266 Doing a power series solution at a regular singular point is almost just like doing it at an ordinary point, *except* that we assume the power series is a product of a run-of-the-mill Taylor series times a factor $(x - x_o)^r$ where r is some constant to be determined. At an ordinary point, you will find $r = 0$. At a regular singular point, you will find an equation for r which is called the *indicial equation*. Once you found r (two of them!), the rest is straightforward—provided $r_1 - r_2$ is not zero or a positive integer. You the get one solution for each r—getting two linearly independent solutions in all.

   The method used here is called the *Frobenius method*.

   - Problems #13 on page 266. Totally straightforward.

9. §5.7 Series Solutions near a Regular Singular Point, Part 2. pp. 267-273. What happens if $r_1 \geq r_2$, both are real, and $r_1 - r_2$ is a zero or a positive integer? Theorem 5.7.1 on page 272 gives the answer. Note: the solution corresponding to the *larger* r_1 is the first solution.

   The procedures for getting the second solution are straightforward but generally very messy. Boyce and Diprima recommends you use the method of undetermined coefficients, which is the intelligent thing to do. You are expected to know that when $r_1 \geq r_2$, both real, the second solution needs special attention, but I don't expect you to remember the details of how to do it. Just remember that it is messy, and it can be found in text books.

- Problems #3 and #7 on page 273. Just the indicial equation and the exponents.

We will skip §5.8, Bessel Equation. Bessel equation is a very important equation, but you now have all the fundamentals to read this section if you should need to know more.