

MAE 306 Notes #6b

Engineering Mathematics II

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Mid Term with Answers

Closed-book Mid-Term; 80 Minutes

Appendix D, Table of Fourier Transforms, is provided on a separate sheet.

Useful Formula

$$\int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) dx = 0, \quad m \neq n \quad (1)$$

$$= \ell, \quad m = n \neq 0, \quad (2)$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) dx = 0, \quad m \neq n \quad (3)$$

$$= \ell, \quad m = n \neq 0, \quad (4)$$

$$\int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) dx = 0, \quad \text{for all } m, n, \quad (5)$$

$$F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) \exp(-i\omega x) dx, \quad (6)$$

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp(i\omega x) dx, \quad (7)$$

Friendly Advice

There are three problems, and you have 80 minutes. Look at all three problems, and budget your time intelligently.

Problems

1. (30 points) The fundamental Green's function $G_\infty(\xi, \tau)$ is defined by:

$$\frac{\partial G_\infty}{\partial \tau} - \frac{\partial^2 G_\infty}{\partial \xi^2} = \delta(\xi)\delta(\tau), \quad (8)$$

where

$$\xi \equiv x - x_o, \quad (9)$$

$$\tau \equiv t_o - t, \quad (10)$$

and $\delta(\xi)$ and $\delta(\tau)$ are the Dirac Delta function with respect to their arguments, respectively (the pulse is located where the argument is zero). The domain of interest is:

$$|\xi| < \infty; \quad \tau \geq 0^-. \quad (11)$$

The initial condition is:

$$G_\infty = 0 \quad \text{at} \quad \tau = 0^-. \quad (12)$$

The boundary condition is

$$G_\infty \rightarrow 0, \quad |\xi| \rightarrow \infty. \quad (13)$$

Find $G_\infty(\xi, \tau)$ using **Fourier Transform**. Use Appendix D provided to help with your algebra and calculus.

2. (40 points) The temperature $u(x, t)$ in a bar of length ℓ is governed by the Diffusion Equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad (14)$$

The boundary conditions are:

$$\frac{\partial u}{\partial x}(x = 0, t) = 0, \quad (15)$$

$$u(x = \ell, t) = 1. \quad (16)$$

The initial condition is:

$$u(x, t = 0) = f(x). \quad (17)$$

- a** Find $u(x, t)$ for $f(x) = 0$. You need not perform the integrations to obtain the Fourier coefficients. Just show the integrals to be performed.
 - b** Describe in words what are the “complications” when $f(x)$ is not zero.
3. (30 points) Here is a first order PDE for u :

$$\frac{\partial u}{\partial x} - \sin(x) \frac{\partial u}{\partial y} = -u^3 \quad (18)$$

The initial condition line is:

$$u(x = 0, y) = \exp(y) \quad \text{on} \quad x = 0; \quad |y| \leq 1. \quad (19)$$

Use $\xi = y$ as a marker on the initial condition line, and use σ as a marker on each characteristics issuing from the initial condition line.

- a** Find $x(\xi, \sigma)$, $y(\xi, \sigma)$ and $u(\xi, \sigma)$.
- b** Sketch on the x, y plane where your solution may be used, and mark clearly where no information is provided by your solution.
- c** Do you have any comments about what interesting things might be happening when x is large and negative?
- d** If you have time, find $\xi(x, y)$ and $\sigma(x, y)$, and with them in hand, write down the analytical formula for $u(x, y)$. Say something helpful to someone who might wish to use your formula.

Answers

The answer given below skipped details that were not found to be a problem. I only present steps that were deemed somewhat difficult by my first scanning of your papers.

In an earlier version posted immediately after the mid-term, there were some typos which are corrected in this version.

Problem #1

We take the Fourier Transform of (8) to obtain an ODE for $\hat{G}_\infty(\omega, \tau)$:

$$\frac{\partial \hat{G}_\infty}{\partial t} + \omega^2 \hat{G}_\infty = \delta(\tau). \quad (20)$$

The Fourier Transform of $\delta(\xi)$ on the right hand side is simply 1. For $\tau > 0^+$, the right hand side is zero. So the solution of this simple ODE is readily obtained:

$$\hat{G}_\infty(\omega, \tau > 0^+) = C_1 \exp(-\omega^2 \tau). \quad (21)$$

We need now to determine C_1 , the integration constant. Integrating (20) from $\tau = 0^-$ to $\tau = 0^+$, we have:

$$\hat{G}_\infty(\omega, \tau = 0^+) = 1. \quad (22)$$

Hence, $C_1 = 1$. Appendix D can now be used to find the inverse Fourier Transform.

1 Problem #2

First split the problem into two parts:

$$u(x, t) = u_\infty(x) + u_i(x, t) \quad (23)$$

where

$$u_\infty(x) = 1 \quad (24)$$

is in fact the “steady” solution after infinite time, and u_i depends on both x and t . The PDE for $u_i(x, t)$ is the same as before,

$$\frac{\partial u_i}{\partial t} = \alpha^2 \frac{\partial^2 u_i}{\partial x^2} \quad (25)$$

but the boundary conditions are now:

$$\frac{\partial u_i}{\partial x}(x = 0, t) = 0, \quad (26)$$

$$u_i(x = \ell, t) = 0. \quad (27)$$

Most importantly, the initial condition is now:

$$u_i(x, t = 0) = f(x) - 1. \quad (28)$$

It is seen that u_i represents the effects of initial condition.

We can now use separation of variables to solve the $u_i(x, t)$ problem. A cosine series should be used. And the separation constants are to be determined by the boundary conditions (26) and (27). And the coefficients of the expansion is determined by the initial condition (28)—taking advantage of the orthogonality of the basis functions.

When $f(x) = 0$, the initial condition is $u_i(x, t = 0) = -1$. When $f(x) \neq 0$, it is no big deal as far as determining the coefficients of the expansion is concerned.

I was taken by surprise that this problem seemed to have caused some difficulties to some of you.

Problem #3

The ODEs along the characteristics are:

$$\frac{dx}{d\sigma} = 1, \quad (29)$$

$$\frac{dy}{d\sigma} = -\sin(x), \quad (30)$$

$$\frac{du}{d\sigma} = -u^3. \quad (31)$$

Integrating and honoring the initial condition on the initial line, we obtain (for $|\xi| \leq 1$):

$$x = \sigma, \quad (32)$$

$$y = \cos(x) - 1 + \xi, \quad (33)$$

$$\frac{1}{2u^2} = \sigma + \frac{1}{2[\exp(\xi)]^2}. \quad (34)$$

Note that the initial line is represented by $\sigma = 0$, and on this line the marker $\xi = y$. The last equation gives:

$$u^2(x, t) = \frac{\exp(2\xi)}{1 + 2\sigma \exp(2\xi)}, \quad |\xi| \leq 1. \quad (35)$$

Note that for negative x (and therefore negative σ , the denominator may vanish. Thus u “blows up” there).

The above solution is valid only for $|\xi| \leq 1$. The graph of the region of applicability is easily sketched by using (33), the boundary of the region being $|\xi| = 1$. So it is a fat cosine band.

Replacing σ and ξ , we have:

$$u(x, t) = \frac{\exp(1 + y - \cos(x))}{\sqrt{1 + 2x \exp[2(1 + y - \cos(x))]}} \quad (36)$$

The danger of this correct analytical result is that people might want to use it in regions of x, y space for which $|\xi| \leq 1$ is not honored.