MAE 306 Notes #11a
Engineering Mathematics II

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1 Reading and Homework Assignments

Study Chapter 23, The Complex Integral Calculus, pp. 1183-1208 of Michael Greenberg's *Advanced Engineering Mathematics*, and do the problem assigned below.

The problems are due on Tuesday, April 27th, 1999, 5PM. Please submit your homework to the MAE 306 homework **IN** tray outside D-302, E.Q.

2 Comments on Readings and Problems

The three big theorems in this chapter are:

The Cauchy Theorm: Theorem 23.3.1 on page 1191.

$$\oint_C f(z)dz = 0. \tag{1}$$

provided $f(z)$ us analytic in a simply-connected region $\mathcal{D}$ and the closed path is piecewise smooth in $\mathcal{D}$.

Fundamental Theorem of the Complex Integral Calculus: Theorem 23.4.1 on page 1197.

$$\int_{z_o}^{z} f(\zeta)d\zeta = F(z) - F(z_o) \tag{2}$$

where $F(z)$ is any primitive of $f(z)$—*i.e.* F'(z)=f(z), and $f(z)$ is analytic in a simply-connected $\mathcal{D}$.

Cauchy Integral Formula: Theorm 23.5.1 on page 1200.

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \tag{3}$$

where $f(z)$ is analytic in a simply-connected $\mathcal{D}$, the closed path is piecewise smooth, and 'a' be any fixed point lying inside the closed path.

In the above, the concept of a "simply connected region" is involved (as defined on bottom of page 826 in Greenberg):

> A domain $\mathcal{D}$ is said to be simply-connected if every closed curve in $\mathcal{D}$ can be shrunk, by a continuous deformation, to any point in $\mathcal{D}$.

For example, in two dimension, the domain outside a circle is not simply-connected, because a closed curve which encloses the circle can not be shrunk to zero.

2.1 Supplementary Notes

Actually, the really foundation of the whole chapter is the *Fundamental Theorem of the Complex Integral Calculus*, (2). Result (1) can be deduced from (2) by considering a closed curve for the integration, and result (3) can be obtained by detailed calculations as performed in Example 3 on page 1185, and repeated in Example 2 on page 1193 (where the integral being worked on is affectionately called *an important little integral.*

Here is my exposition on this subject. Any (complex) function $F(x, y)$ can, in general, be written in terms of z and $\bar{z}$. We limit ourselves only to (complex) functions which depend only on z, and NOT $\bar{z}$—*e.g.* $F = F(z)$. If $F(z)$ exists and is differentiable in a certain domain, then $F(z)$ is said to be analytic in that domain.

Consider now the vector ∇F, the gradient vector of F. The differential dF along a differential (real) displacement (vector) $d\mathbf{s} = \mathbf{e}_x dx + \mathbf{e}_y dy$ is:

$$dF = \nabla F \cdot d\mathbf{s} \tag{4}$$

where ∇F is the gradient of F and $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors in x, y directions, respectively. It is immediately clear from (4) that:

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} \nabla F \cdot d\mathbf{s}. \tag{5}$$

Most importantly, it is seen that the value of this integral does not change as one "deforms" the path of integration, so long as the terminal points are held fixed. We shall extensively exploit this path-independence result.

By chain-rule, we have:

$$\nabla F = \frac{dF}{dz}(\mathbf{e}_x + i\mathbf{e}_y). \tag{6}$$

Using this and $d\mathbf{s} = \mathbf{e}_x dx + \mathbf{e}_y dy$ om (7), we have:

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(z)dz. \tag{7}$$

where $f(z) = dF(z)/d(z) = F'(z)$. So, we have "derived" (2)—when the starting and ending points z_1 and z_2 are distinct.

If the starting point and the ending points are immediately adjacent to each other (they are the same point), and the integration path specified can be "deformed" and "shrunk" into nothingness, and if the function $F(z)$ is single-valued everywhere on the integration path (*including* the tiny gap between the starting and the terminal points), then we have "derived" (1).

If the closed-loop integration path of interest includes the point $z = a$ "inside," then a simple generalization of Example 3 on page 1185 and Example 2 on page 1193 for the case $n = -1$ will yield the result displayed in (3).[1]

So, we have provided "proofs" for these three important theorems. They are enormously powerful and useful theorems!

3 Comments and Problems

§23.1, 23.2: Just scan the material before Example 3 on page 1185 and study this example with care. Look at the answers when $n \neq -1$: the answer is ZERO, and is independent of the value of R (the radius) and n (provided it is not -1). What happens when $n = -1$ precisely? The answer is $2\pi i$—it is independent of R.

In §23.2.2, the upper "bound" of a line integral is studied. The material is pretty straightforward. In our latter work next week, we shall

[1]For the $n = -1$ case, it turns out that the primitive $F(z) = \ln z$ which is not single-valued unless a branch cut is imposed. For other values of integer n, the corresponding $F(z)$ are single-valued. If n is not an integer, then we have quite a mess.

take advantage of the skills learn here to estimate the bound of some segments of a line integral of a closed curve.

- Problem 1d on page 1188. Hint: You are given the path to integrate. Try to deform the path to make life simple for yourself.
- Problem 2 on page 1188. This problem shows you by an example that the line integral $\int_{z_1}^{z_2} f(\bar{z})dz$ is path dependent. So, all you need to do is to pick two different paths (east first then north, or north first then east), and perform the desired integrations. You will get two different answers.

§23.3: Scan Example 2, the *important little integral*, and Example 3, an application of this result.

- Problem 4d, 4f and 4g on page 1194

§23.4: Theorem 23.4.1 is the grand-daddy of them all.

- Problems 3a, 3b and 3d on page 1198. This is essentially an excercise in integration.

§23.5: Theorem 23.5.1 is one of the big three theorems in this chapter! Study Example 2. Take note of the **generalized Cauchy Integral formula**, equation (22) at the bottom of page 1203. Study Example 3 and 4 on page 1204 for its applications.

- Problems 1a and 1c on page 1205. Hint: find the "poles" first.