

# MAE 306 Notes #12a

## Engineering Mathematics II

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### 1 Reading and Homework Assignments

Study Chapter 24, Taylor Series, Laurent Series, and the Residue Theorem, pp. 1209-1259 of Michael Greenberg's *Advanced Engineering Mathematics*, and do the problem assigned below.

The problems are due on Tuesday, May 4th, 1999, 5PM. Please submit your homework to the MAE 306 homework **IN** tray outside D-302, E.Q.

Some of the problems "assigned" I only ask you to "study" them. You need not do them.

### 2 Comments and Problems

§24.1, 24.2: I assume all of you have studied infinite series before, so the materials here are familiar to you. The focus is on the Taylor Series (displayed as (10) on page 1212). The main result is Theorem 24.2.6, *power series convergence in a disk*. The "proof" given on page 1214 is wordy and inelegant. The message is: a Taylor series of a complex variable is convergent in a disk, and the  $f(z)$  it represents is analytic in the disk.

The concept of *analytic continuation* expounded on page 1218 is important. If you do another Taylor series about a point in your first disk of convergence, the new Taylor series will converge in some new territory beyond the first disk. You can repeat this process, and come up with newer Taylor series which will converge in more additional territories.

For unknown reasons, Greenberg does not state the main result cleanly. I would state the main result as follows:

If the Taylor Series of two functions  $f(z)$  and  $g(z)$  are identical in a finite domain  $\mathcal{D}$ , then the analytic continuation of  $f(z)$  is identical to the analytic continuation of  $g(z)$  for all  $z$ . Note: the theorem did not say the original  $f(z)$  and  $g(z)$  are identical for all  $z$ ; only their analytic continuations.

Greenberg essentially says the same thing with his Theorem 24.2.9. I will explain in class that this theorem is the rationalization used for not respecting the  $s > 0$  admonitions in Appendix C, the table of Laplace Transforms on page 1271.

- Problem 9 on page 1224. Remember, the region of convergence is a circular disk, and it must not enclose any singularity.

**§24.3:** Laurent Series. What happens if you want an infinite series representation of a function in an annulus region centered about  $z = a$ ? The answer is Theorem 24.3.1, which also provides explicit formula (3) on page 1225 for the coefficients  $c_n$ 's. An useful hint: few, if any, ever uses formula (3) to evaluate the Laurent Series coefficients! The theorem tells us the "form" of Laurent Series. In practice, the coefficients are nearly always obtained by algebraic tricks. You need to know the Binomial expansion formula (see Problem 12 on page 1224):

$$(1+x)^\mu = 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!}x^3 + \dots \quad (1)$$

- Problem 2 on page 1233. You need the Binomial formula above.
- Problem 4a on page 1233; the center of the annulus is at  $z = i$ .
- Problem 4d on page 1233. Hint: add and subtract  $1/z$  to  $1/(\exp(z)-1)$ , and concentrate on  $g(z) = 1/(\exp(z)-1) - 1/z$ . Is  $g(z)$  analytic in the neighborhood of  $z = 0$ ? If so, does it have a Taylor Series?

**§24.4:** Note the definition of *isolated singularity*. Example 1 is an example of non-isolated singularity.

- Study Example 1 on page 1234 to see an example of nonisolated singularity.

**§24.5:** The Residue Theorem! This is the heart of the chapter. Actually, we already know this theorem, previously known as the Cauchy Integral Formula. The Residue Theorem is just a dressed-up version of the Cauchy Integral Formula. Greenberg gives 9 examples in this section (between page 1242 and page 1255. We will do several of them in class.

- Problem 1a on page 1255.
- Problem 2a and 2d on page 1255.
- Problem 6a and 6b on page 1256. Note:  $x^{a-1}$  with  $0 < a < 1$  needs a branch cut to remain single-valued. Hint for 6a:  $(-1)^{a-1} = \exp(i\pi(a-1)) = -\exp(i\pi a)$ .
- Problem 8b on page 1256. Note:  $\ln z = \ln r + i\theta$  needs a branch cut to remain single-valued.
- Problems 11a on page 1257. There is a typo in my version of Greenberg's (11.1): under the limit sign, it should have  $\epsilon_1 \rightarrow 0$  and  $\epsilon_2 \rightarrow 0$ . The big deal is: the "principal value" of an integral and includes a singular point is the integral skipping the singular point by **equal distance** on both sides in the limit when the skipping gap shrinks to zero.
- Study Problem 12 on page 1257 and understand what is being done there. Three tricks were used. First, replace  $\sin x$  by  $\exp(iz)$ , second, select the closed loop as shown in diagram, third, use the inequality  $\sin x \geq 2x/\pi$  for  $0 \leq x \leq \pi/2$  to show that the big semi-circle contribution is zero. There is a typo on top of page 1258: it is "section 23.2" instead of "24.2."