

MAE 306 Notes #13a

Engineering Mathematics II

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You are welcome to ask questions by email (lam@princeton.edu), or call (8-5133) to make an appointment and then come by to ask questions in person.

1 Study Hints for the Final

- The final exam covers all the materials of the course: Chapters 17 through 24, plus my supplementary notes on first order PDEs (Notes 5a). The primary topics will be those studied after the mid-term: Chapters 19 through 24. But I may ask questions about materials in the first half of the course also.
- Initial and boundary conditions. We have dealt with three kinds of second order linear PDEs: parabolic (diffusion), hyperbolic (wave) and elliptic (Laplace). You should know the classification via $B^2 - AC$ by heart. You may want to take a look at Notes 7b-§2 to refresh yourself on the concept of characteristics. Most importantly, you should know that the type of initial/boundary conditions for these three types of PDEs are different. For the diffusion equation, the independent variable t has a preferred direction; it is OK to predict the future mathematically, but it is not OK to compute for the past! The domain of interest is always “open” in the future direction. You need one initial condition (at $t = 0$) and one boundary condition at the boundaries. For the wave equation, the equation itself does not change when you reverse the direction of the time arrow, so the investigator (you) have to enforce

casualty personally). You need TWO initial conditions on a time-like domain boundary, and one boundary condition at a space-like boundary. For the Laplace equation, all independent variables are space-like. Solution can not be constructed by a “marching scheme” as were the case for the other two PDEs.

- You should be generally informed about the special features of numerical methods for each of the three types of PDEs. In particular: how do you pick the size of Δt for diffusion equations? For wave equations?
- Study Notes 7c on how to exploit superposition in solving these linear problems. Certainly, there will be a generic problem of this kind in the final.
- You should be generally informed on curvilinear coordinates (see Notes 8a). If you ever need to find $\text{grad } u$, $\text{Div } \mathbf{V}$, $\text{Curl } \mathbf{V}$, or $\nabla^2 u$, you should know what books to look up. (You don't need to memorize these formula except to know that they exist in some books).
- Study Notes 8b. This is very similar to the stuff in Notes 7c, except that the concept of eigenvalues and eigenfunctions are brought in more explicitly. The materials in Notes 8b on Green's functions, etc (§3, 4, 5, 6) will not be in the final exam.
- Then we studied Theory of a Complex Variable $z = x + iy$, starting with Chapter 21 which gave you general ability in manipulating complex numbers and functions. You must know that $F(z)$ is a special class of complex function involving x and y —it is special because its dependence of \bar{z} is not allowed. You must know the derivation of the Cauchy-Riemann condition. Look over the buzz words on page 7 of Notes 9a. You are expected to know them by heart.
- Conformal Mapping. Given a relation $w = F(z)$, any geometric curve in the z -plane can be mapped into the w -plane. You need to know the derivation of the fact the the mapping is conformal whenever $F'(z) \neq 0$, and what the definition of conformality is. You may be given a specific $F(z)$ and be asked what happened to a curve in the z -plane after it is mapped into the w plane. You may be asked to explain why conformal mapping provides a tool for finding solutions to the two-dimensional Laplace equation.

- The Complex Integral Calculus
- Taylor and Laurent Series and the Residue Theorem. Using the elegant and simple result of the Cauchy's Theorem and the knowledge that the value of an line integral in the complex plane is unchanged when the path is deformed, we arrived at the Residue Theorem—which essentially said let us just deform the contour until we wrap around the poles. Now, remember, the only contribution comes only from the “simple” poles. What happens if you are wrapping around a N -order pole? You get the Laurent Series about that pole! How do you find the Laurent Series? Almost never use the formula derived in the Laurent Series Proof. You use one or the other kinds of tricks. The Binomial Theorem often plays a role. You should know the Binomial Theorem by heart, but in the final I will give it to you if it is needed to do the problems.
- Branch cut is a big deal. You can be certain that there will be a contour integration problem involving a branch cut.
- The concept of analytical continuation must be understood. How come you are messy around in the left half plane of s when you are doing Laplace Transform inversion? Sir! you reply, I am not messy around with the honest-to-goodness $F(s)$, only with its analytical continuation, sir!

2 Inverse Laplace Transform

One of you asked me after class how was the formula for the inverse Laplace Transform “derived.” Here is an answer—showing the role played by the theory of complex variables.

The Laplace Transform of $f(t)$, denoted by $F(s)$, is defined by:

$$F(s) = \int_0^{\infty} f(t') \exp(-st') dt', \quad (1)$$

provided the real part of the complex variable $s > \gamma$ where γ is sufficiently positive to ensure the convergence of the integral. Note: s is considered a complex variable, and $F(s)$ is undefined when $s < \gamma$

To look for the inverse Laplace Transform, we proceed as follows. We formally express the inverse Laplace Transform in terms of a contour integral in the complex s plane:

$$f(t) = \int_C \kappa(s, t) F(s) ds \quad (2)$$

where $\kappa(s, t)$ is an unknown “kernel” and C is an unknown path of integration, both are to be found. Substituting for $F(s)$ using (1) into (2), we obtain:

$$f(t) = \int_0^\infty \left\{ \int_C \kappa(s, t) \exp(-st') ds \right\} f(t') dt' \quad (3)$$

Obviously, for this equation to be valid, the curly bracket in the integrand must be our old friend, the Dirac Delta Function.¹

$$\delta(t - t') = \int_C \kappa(s, t) \exp(-st') ds. \quad (4)$$

Our task now is to find $\kappa(s, t)$ and the integration path C so that (4) is happy.

We now exploit our knowledge of Fourier Transform, particularly entry 10 of Appendix D. For $\kappa(s, t)$, it is clear that we should choose:

$$\kappa(s, t) = \frac{1}{2\pi i} \exp(st) \quad (5)$$

For the path C , it is clear that the line integral should go from $s = \gamma - i\infty$ to $s = \gamma + i\infty$.

¹Remember, the Dirac Delta Function is wonderful; whenever it shows up in the integrand, the answer of the integration is obvious!