

MAE 306 Notes #1b

Engineering Mathematics II

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1 Reinventing Fourier Series

What if we have never heard of Sines and Cosines? What if we do not know (24a,b,c) in §17.3 of Greenberg's book? What if we have lost or have forgotten Euler's Formula (5b,c,d) in §17.3 of Greenberg's book?

How can we be convinced that the more terms included in a Fourier Series, the "better" the results will be?

Are sines and cosines the only game in town? Are there other set of functions that can do what sines and cosines do for us in Fourier Series?

1.1 The Operator, its Adjoint, and Self-adjointness

Consider the linear differential operator \mathcal{L} defined below:

$$\mathcal{L} \equiv -\frac{1}{w} \frac{d}{dx} \left(p \frac{d}{dx} \right) \quad (1)$$

where, for the sake of simplicity, we confine our attention to $w = +1$ and $p = +1$. The expression $\mathcal{L}(u)$ means *operate on $u(x)$ in accordance to instructions given in this definition—e.g. the result in this case is simply $\mathcal{L}(u) = -u''(x)$.*

We now define the inner product of two functions $u(x)$ and $v(x)$ in the interval $-\pi \leq x \leq \pi$:

$$\langle u, v \rangle \equiv \int_{-\pi}^{\pi} u(x)v(x)w dx. \quad (2)$$

Note that $\langle u, u \rangle$ is always positive (unless $u(x)$ is identically zero—the uninteresting trivial case).

Now consider only those functions (at least twice differentiable) $u(x)$ and $v(x)$ which satisfy the following “boundary” conditions and are otherwise arbitrary:

$$u(-\pi) = u(\pi) = v(-\pi) = v(\pi) = 0. \quad (3)$$

We have:

$$\langle \mathcal{L}(u), v \rangle = - \int_{-\pi}^{\pi} \frac{d^2 u}{dx^2} v(x) dx \quad (4)$$

where we have replaced p and w by $+1$ for the simple special operator under study. Integrating by parts, we obtain:

$$\langle \mathcal{L}(u), v \rangle = - \left[\frac{du}{dx} v(x) \right]_{x=-\pi}^{x=\pi} + \int_{-\pi}^{\pi} \frac{du}{dx} \frac{dv}{dx} dx \quad (5)$$

The first term on the right hand side vanishes because of (3). Integrating by parts once more, we obtain:

$$\langle \mathcal{L}(u), v \rangle = - \left[u(x) \frac{dv}{dx} \right]_{x=-\pi}^{x=\pi} + \int_{-\pi}^{\pi} u(x) \mathcal{L}^*(v) dx \quad (6)$$

where \mathcal{L}^* is called the *adjoint* of \mathcal{L} and is given by (for this problem)

$$\mathcal{L}^*(v) = - \frac{d^2 v}{dx^2}. \quad (7)$$

Since the first term in (6) vanishes again because of (3), we have:

$$\langle \mathcal{L}(u), v \rangle = \langle u, \mathcal{L}^*(v) \rangle \quad (8)$$

For the \mathcal{L} operator under consideration, we find that $\mathcal{L} = \mathcal{L}^*$ —the original operator and its adjoints are the same. Such operators are called *self-adjoint* operators, and for such operators the so-called *Lagrange Identity* (page 898 of Greenberg) follows from (8) immediately:

$$\langle \mathcal{L}(u), v \rangle = \langle u, \mathcal{L}(v) \rangle, \quad (\mathcal{L} \text{ self-adjointed assumed}) \quad (9)$$

It goes without saying that both $u(x)$ and $v(x)$ are assumed to honor the boundary conditions in all of the above developments.

1.2 Eigen stuffs

Let us choose $u = v$. we have, for the operator we chose to study,

$$\langle \mathcal{L}(u), u \rangle = \int_{-\pi}^{\pi} \left(\frac{du}{dx} \right)^2 dx \geq 0. \quad (10)$$

Now, consider the scalar $\lambda(u)$ defined by:

$$\lambda(u) \equiv \frac{\langle \mathcal{L}(u), u \rangle}{\langle u, u \rangle}. \quad (11)$$

Note that $\lambda(u)$ is a scalar whose value depends on the $u(x)$ you have chosen to be worked on by the right hand side. It is clear that λ is always positive. An operator \mathcal{L} is said to be *positive-definite* if λ is positive-definite using any $u(x)$.

We now ask the question: does λ have a lower bound? An alternative way of asking the same question is: what kind of $u_o(x)$ will give us the smallest λ ? Let us rewrite (11) as follows:

$$\langle \mathcal{L}(u), u \rangle = \lambda \langle u, u \rangle. \quad (12)$$

We now look in the neighborhood of $u_o(x)$ by defining:

$$u(x) = u_o(x) + \delta u \quad (13)$$

where δu is the deviation of $u(x)$ from the $u_o(x)$ which gives the smallest $\lambda = \lambda_o$. The λ associated with $u(x)$ will be denoted by $\lambda_o + \delta\lambda$. We can now plug everything in, and look for the condition on $u_o(x)$ which would yield $\delta\lambda = 0$ for small but arbitrary δu . The algebra involved is pretty straightforward. The answer is:

$$\mathcal{L}(u_o) = \lambda_o u_o. \quad (14)$$

For our chosen simple \mathcal{L} , this equation is, in long hand,

$$\frac{d^2 u_o}{dx^2} + \lambda_o u_o = 0. \quad (15)$$

It is helpful to recall that the boundary conditions for u_o is

$$u_o(-\pi) = u_o(\pi) = 0. \quad (16)$$

The ODE problem posed by (15) and (16) is called an *eigenvalue* problem: one needs to find both the value of λ_o and $u_o(x)$. Note that the trivial solution $u_o(x) = 0$ is not useful and is discarded. Only non-trivial solutions are of interest here.

For our problem, the solution of the eigenvalue problem is easily obtained:

$$\lambda_o = n^2, \quad n = 1, 2, \dots; \quad (17)$$

$$u_o(x) \propto \sin(nx). \quad (18)$$

In other words, the absolute smallest value for λ is $\lambda_o = 1$, obtained when $u = u_o(x) \propto \sin(x)$.

Since $u_o(x)$ and λ_o depend on n , we shall replace the o subscript by an integer subscript: $u_n(x) = \sin(nx)$ and $\lambda_n = n^2$. Note: the number of eigenvalues, λ_n , is infinite, and there is no upper bound for the magnitude of λ_n as n increases. In particular, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

1.3 Orthogonality of the eigenfunctions

We write down the ODE for $u_i(x)$ and $u_j(x)$ as follows:

$$\mathcal{L}(u_i) = \lambda_i u_i, \quad (19)$$

$$\mathcal{L}(u_j) = \lambda_j u_j \quad (20)$$

where i and j are integers. We now take the inner product of (19) with u_j , and subtract from it the inner product of u_i with (20). Taking advantage of the Lagrange Identity (because we are dealing with a self-adjoint operator), we obtain:

$$(\lambda_i - \lambda_j) \langle u_i, u_j \rangle = 0. \quad (21)$$

Whenever $i \neq j$, *i.e.* when $u_i(x)$ and $u_j(x)$ are distinct, the above equation says their inner product is zero. In other words, they are *orthogonal*. This result is completely consistent with the result found in any integral tables and in Greenberg:

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = 0, \quad m \neq n. \quad (22)$$

It is obvious that $\langle u_n, u_n \rangle$ is never zero for any non-trivial $u_n(x)$. Conventionally, the $u_n(x)$'s are usually scaled so that $\langle u_n, u_n \rangle = 1$ is honored.¹

¹It is important to note that u_i and u_j do not obey (21) if \mathcal{L} is not self-adjoint. In other words, u_i and u_j are not orthogonal unless \mathcal{L} is self-adjoint.

1.4 Euler's Formulas for the coefficients

Now consider any odd function $f(x)$ in the interval $-\pi \leq x \leq \pi$. We define $F_N(x)$ to be a N term series as follows to represent $f(x)$:

$$F_N(x) \equiv \sum_{n=1}^N b_n u_n(x); \quad u_n(x) \equiv \sin(nx). \quad (23)$$

The deviation between $f(x)$ and $F(x)$ is:

$$e_N(x; b_n) = f(x) - F_N(x). \quad (24)$$

The total mean-square error over the interval is the 2-norm of e_N :

$$E_N(b_n) \equiv \langle e_N, e_N \rangle \quad (25)$$

How should the coefficients b_n 's be chosen so that $E(b_n)$ is a minimum with respect to the b_n 's? We find the minimum by setting $\partial E_N / \partial b_n = 0$:

$$\frac{\partial E_N}{\partial b_n} = 2 \langle u_n, e_N \rangle = 0, \quad n = 1, 2, \dots, N, \quad (26)$$

yielding (taking full advantage of orthogonality),

$$b_n = \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle}. \quad (27)$$

You can easily verify that this choice of b_n 's is precisely what was recommended to us by Euler (see (24b) on page 847 of Greenberg)! In other words, when this b_n formula is used, $F_N(x)$ is precisely the N -term Fourier Sine Series representation of $f(x)$. In other words, we have just reinvented the Fourier Sine Series, and provide the guarantee that E_N is a minimum when Euler formulas are used for the evaluation of the coefficients!

1.5 Why does the representation improve when more terms are included?

We now consider what happens if we use $u = e_N$ to compute the resulting $\lambda(e_N)$. We have:

$$\lambda(e_N) = \frac{\langle \mathcal{L}(e_N), e_N \rangle}{\langle e_N, e_N \rangle}. \quad (28)$$

We know $\lambda(e_N) > \lambda_1 > 0$ because our \mathcal{L} is positive-definite and $\lambda_1 = 1$ is the absolute minimum. In fact, it is possible to prove in general that $\lambda(e_N) > \lambda_N$ always, so that the theoretically smallest value for $\lambda(e_N)$ is λ_{N+1} , or $(N+1)^2$ for our problem. The formal proof is a bit long winded, but the validity of the claim should be intuitively obvious.

Hence, we have:

$$E_N = \langle e_N, e_N \rangle = \frac{\langle \mathcal{L}(e_N), e_N \rangle}{\lambda(e_N)} \quad (29)$$

$$\leq \frac{\langle \mathcal{L}(e_N), e_N \rangle}{(N+1)^2}. \quad (30)$$

Both the numerator and the denominator on the right hand side now depends on N . Let us compute the numerator $\langle \mathcal{L}(e_N), e_N \rangle$:

$$\begin{aligned} \langle \mathcal{L}(e_N), e_N \rangle &= \langle \mathcal{L}(f - F_N), f - F_N \rangle \\ &= \langle \mathcal{L}(f), f \rangle - \langle \mathcal{L}(f), F_N \rangle - \langle \mathcal{L}(F_N), f \rangle \\ &\quad + \langle \mathcal{L}(F_N), F_N \rangle. \end{aligned} \quad (31)$$

Using the Lagrange Identity, the middle two terms on the right hand side are the same and can be combined:

$$\langle \mathcal{L}(e_N), e_N \rangle = \langle \mathcal{L}(f), f \rangle - 2 \langle \mathcal{L}(F_N), f \rangle + \langle \mathcal{L}(F_N), F_N \rangle.$$

Replacing the f in the middle term by $e_N + F_N$, we obtain:

$$\langle \mathcal{L}(e_N), e_N \rangle = \langle \mathcal{L}(f), f \rangle - 2 \langle \mathcal{L}(F_N), e_N \rangle - \langle \mathcal{L}(F_N), F_N \rangle.$$

Using

$$\mathcal{L}(F_N) = - \sum_{n=1}^N \lambda_n b_n u_n(x) \quad (32)$$

and $\langle u_n, e_N \rangle = 0$ according to (26), we can conclude that the middle term is precisely zero. We now have:

$$\begin{aligned} \langle \mathcal{L}(e_N), e_N \rangle &= \langle \mathcal{L}(f), f \rangle - \langle \mathcal{L}(F_N), F_N \rangle \\ &\leq \langle \mathcal{L}(f), f \rangle. \end{aligned} \quad (33)$$

The inequality hold because $\langle \mathcal{L}(F_N), F_N \rangle$ is positive-definite because \mathcal{L} is positive-definite.

Finally, (30) becomes:

$$E_N = \langle e_N, e_N \rangle \leq \frac{\langle \mathcal{L}(f), f \rangle}{(N+1)^2}, \quad (34)$$

which provides an upper bound for E_N . Note that the numerator on the right hand side is independent of N . So the larger N is, the smaller E_N shall be.

This completes the proof that the more terms you include in your series, the better is your representation—provided you did not miss any eigenfunctions. For example, a 3000 term Series with $u_2(x)$ missing in the line up is not guaranteed to be better than a 3 term Series with the same $u_2(x)$ missing. If no eigenfunctions are missing from the line up, then the more terms the better.

1.6 Closing Remarks

If the following boundary conditions

$$\frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi) = \frac{dv}{dx}(-\pi) = \frac{dv}{dx}(\pi) = 0 \quad (35)$$

had been used instead of (3), the set of eigen functions found would have been $u_n = \cos(nx)$ instead of $\sin(nx)$. We would then have reinvented the Fourier Cosine Series. The rest of the Euler's formula on page 857 of Greenberg can then be derived.

When \mathcal{L} is in the following form:

$$\mathcal{L} = -\frac{1}{w} \left[\frac{d}{dx} \left(p \frac{d}{dx} \right) + q \right] \quad (36)$$

where $w(x) > 0$ and $p(x) > 0$ in the interval of interest ($a \leq x \leq b$), the problem is said to be a Sturm-Liouville Problem. It is easily verified that the Sturm-Liouville linear differential operator is self-adjoint. The general boundary condition is:

$$\alpha u(a) + \beta \frac{du}{dx}(a) = 0, \quad (37)$$

$$\gamma u(b) + \sigma \frac{du}{dx}(b) = 0 \quad (38)$$

where $\alpha, \beta, \gamma, \sigma$ are constants. When $p(x)$, and $w(x)$ are not constants and $q(x)$ is not identically zero, the eigen functions $u_n(x)$'s are not sines and cosines, and the eigenvalues λ_n 's are more complicated functions of n than the case studied here. Each Sturm-Liouville problem can generate its own set of eigenfunctions (freeing ourselves from the tyranny and dictatorship of sines and cosines), bringing in many new players in town, such as Bessel functions. But all theoretical developments proceed analogously.

The spectacular result here is that $F_N(x)$, which is made up from adding a lot of twice differentiable continuous functions together, can represent any $f(x)$ in the interval of interest, even if $f(x)$ is piece-wise continuous. In other words, even if $f(x)$ has a finite number of discontinuities. The claim is that $F_N(x)$, in the large N limit, will converge to the $(f(x_o^+) + f(x_o^-))/2$ where $x = x_o$ is the point of discontinuity. The proof is sketched below.² The theories on Gibb's phenomena in the neighborhood of discontinuities are too long to be included here.

²If there is a discontinuity of $f(x)$ at some point (e.g. $x=0$), split $f(x)$ into the sum of a completely continuous function $g(x)$ and a *step-function* (denoted by $H(x)$) with the discontinuous step located at $x = 0$. Now do the Fourier Series for both. Of course, there is no problem with $g(x)$ which now has no discontinuity. Now, further split $H(x)$ into an even ($H_e(x)$; *it is a constant*) and an odd ($H_o(x)$) function. Use a sine series for $H_o(x)$. It is now clear that the value of the Fourier Series for $H_e(x)$ at $x = 0$ will converge to zero, while $H_e(0^-) = -H_e(0^+) \neq 0$.