

MAE 306 Notes #2a

Engineering Mathematics II

S. H. Lam

February 9, 1999

1 Reading and Homework Assignments

Study the remainder of Chapter 17, pp. 887-942 of Michael Greenberg's *Advanced Engineering Mathematics*, and do the problems assigned below.

The problems are due on Tuesday, February 16, 1999, 5PM. Please submit your homework to the MAE 306 homework **IN** tray outside D-302, E.Q.

1.1 Some backgrounds

Unless explicitly defined otherwise, the squareroot of -1 shall be denoted by i . Hence, by definition, we have $i^2 = -1$, and $i^3 = -i$, etc. The symbol i is called the imaginary number.

Instead of dealing only with real numbers, we now include "complex numbers" in our considerations. A complex number (variable or function) consists of two parts: a real part and an imaginary part.

The most important identity is:

$$\exp(i\xi) = \cos(\xi) + i \sin(\xi), \quad \xi \text{ is real.} \quad (1)$$

In other words, $\exp(i\xi)$ is a complex function, its real part is $\cos(\xi)$ and its imaginary part is $i \sin(\xi)$.

It is easy to show from (1) that

$$\cos(\xi) = \frac{\exp(i\xi) + \exp(-i\xi)}{2}, \quad (2)$$

$$\sin(\xi) = \frac{\exp(i\xi) - \exp(-i\xi)}{2i}, \quad (3)$$

These relations are very similar to the relations involving the so-called *hyperbolic functions* (which are always real functions):

$$\cosh(\xi) = \frac{\exp(\xi) + \exp(-\xi)}{2}, \quad (4)$$

$$\sinh(\xi) = \frac{\exp(\xi) - \exp(-\xi)}{2}, \quad (5)$$

$$\exp(\xi) = \cosh(\xi) + \sinh(\xi). \quad (6)$$

1.2 Comments on the readings

- Read Chapter 17 Review on pp. 940-942 first to get an overview of the whole chapter.
- §17.7. In my Notes 1b, I gave an exposition on the Sturm-Liouville Theory on the problem posed by (1a) and (1b) on page 887 of Greenberg for the simple case $p = 1$, $w = 1$, $q = 0$, $\alpha = 0$ and $\gamma = 0$. We were able to reinvent all the wonderful properties of Fourier Series. Essentially, the same exposition can be done in the more general case—arriving at the same conclusions (when only $p(x) > 0$ and $w(x) > 0$ are imposed). The important concepts introduced and used are: the linear differential operator \mathcal{L} , the adjoint operator \mathcal{L}^* , self-adjointness ($\mathcal{L} = \mathcal{L}^*$), positive-definiteness ($(\mathcal{L}(u), u) > 0$) of the operator.

Problems on pp.902-905:

- #1, (c). A straightforward problem The eigen functions are cosines.
- #3. The Sturm-Liouville operator is, by definition, self-adjoint. Not all linear, second order differential operators are self-adjoint. This problem shows that, with very minor restrictions, (nearly) any linear second order differential operator can be “recasted” into a self-adjoint form. Hint: use an integration factor!

So far, whenever we encountered an eigenvalue problem (such as problem #1(c)), we had always been lucky—we always knew how to find the exact analytical solutions of the ODE, and were able to find the eigenvalues analytically also (by honoring the boundary conditions). What happens when the given self-adjoint ODE (if it was not given in

self-adjoint form, we just learned how to make it self-adjoint by doing problem #3 assigned above) has no known analytical solutions? For example, what happens if we have to solve the following eigenvalue problem:

$$y'' + \lambda w(x)y = 0, \tag{7}$$

$$y(-\pi) = y(\pi) = 0. \tag{8}$$

and $w(x) > 0$ is a very messy and complicated function of x , so messy and complicated that you are certain no analytical solution is known anywhere in the world, including Mathematica and Maple and other similar softwares?

An obvious answer to this question is, obviously, “use numerical computations!” But how? I will discuss this in class.¹

- §17.8. Just scan this section dealing with “singular” and periodic Sturm-Liouville problems. The punch line is near the bottom of page 991: “Essentially, the upshot is that ‘all is well’: we still ... over the (a,b) interval.”

Problems on pp.912-913:

- #6. I am NOT assigning this as a homework problem to *do*. I am just suggesting you to *read* the problem, and learn the word “Chebyshev.” The Chebyshev functions as displayed in (6.4) on page 912 is used extensively in many engineering problems. While you are at it, take a look also at §4.4 on page 212. You will see the Legendre Equation (which is self-adjoint); its eigenvalues and eigenfunctions are given on page 214.

- §17.9, Fourier Integral. We now consider $f(x)$ in the whole $-\infty \leq x \leq \infty$ domain, with the most important new restriction that $\int_{-\infty}^{\infty} |f(x)| dx$ be convergent. The useful formula here is (2a) and (2b) on page 914-915. The Theorem of the week is Theorem 17.9.1 on page 915.

Problem on p.919:

- #2(b),
- #2(c),

¹If you can't find the real McCoy, what is the next best thing you can think of to do?

– #2(d).

- §17.10. Fourier Transform! It is a blood relative of Laplace Transform that you learned and loved in MAE 305. The major equations are (6a) and (6b) on page 921. Note: except for the difference in the factor in front of the integrals, the integrands of the two integrals are VERY similar—watch the signs on the two exponentials! Substituting $f(x)$ as given by (6b) into (6a), I obtain:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x') \exp(-i\omega x') dx' \right) \exp(i\omega x) d\omega. \quad (9)$$

where I had been careful to use x' as the dummy integration variable inside the big bracket which is $\hat{f}(\omega)$. As it now stands, the x' integration is to be carried out first, then the ω integration is to be carried out later. Lets switch the order. We now have:

$$f(x) = \int_{-\infty}^{\infty} \Lambda(x, x') f(x') dx' \quad (10)$$

where

$$\Lambda(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega(x - x')) d\omega \quad (11)$$

It is clear, by inspection of (10), that $\Lambda(x, x')$ is an interesting function: if you multiply it by $f(x')$ and integrate over all x , you recover $f(x)$ back!!! Equation (11) above is the definition of the so-called *Dirac Delta Function*, usually denoted by $\delta(x - x')$ (instead of $\Lambda(x, x')$). In English, $\delta(\xi)$ is identically zero whenever $\xi \neq 0$, but it is infinitely large at $\xi = 0$; the integral over this “Washington Monument” looking function is unity. (What is $\hat{f}(\omega)$ if $f(x) = \delta(x)$?) When we dive into Theory of Complex Variables later, we shall be able to deal with this kind of integral more easily. For now, you have to take my words (and Greenberg’s) for it.

Study the examples on page 921-929. In particular, study example 7 to see how to exploit the “convolution property” of Fourier Transforms. Unfortunately, I found a number of typos on pages 925 and 926, and these typos can make your study of the examples on these pages very painful. Here is a list of the ones I found:

– “entry 10” just below (26) on page 925 should be “entry 11.”

- “entry 9” just below (28) on page 926 should be “entry 11.”
- “entry 16” just below (29) on page 926 should be “entry 17.”

There may be more; so be cautious as you read the book.

Problem on pp.932-934: These problems are for you to practice using (6a) and (6b) on Greenberg’s page 921, and to practice using Appendix D—following the methodology of the examples.

- #4(b).
 - #4(f).
 - #6(c).
 - #6(m).
 - I suggest you READ (not DO), problem 11. It shows you one way of “extending” Appendix D.
- Read Chapter 17 Review at the end of the chapter, again.