MAE 306 Notes #3a
Engineering Mathematics II

S. H. Lam

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1 Reading and Homework Assignments

Study Chapter 18, pp. 943-1016 of Michael Greenberg's *Advanced Engineering Mathematics*, and do the problems assigned below.

The problems are due on Tuesday, February 23, 1999, 5PM. Please submit your homework to the MAE 306 homework **IN** tray outside D-302, E.Q.

1.1 Comments on the readings

- Read Chapter 18's Review on pp. 1055-1057 first to get an overview of the whole chapter.

- §18.1-§18.2. Equation (10) on page 946 is the most general form of a linear, second order PDE with ONE dependent variable u two independent variables x and y. In my lecture, I showed the derivation of the classification as displayed by (11) on page 947, and showed that the same classification remains valid when the PDE is "quasi-linear," a more general condition than just "linear."

  If we replace y in favor of t, the strictly linear parabolic (diffusion) PDE is usually in the form of (21) on page 949 where α^2, V and H may be functions of x and t. A sample derivation of this kind of PDE is given on page 948-949 in §18.2.3. The typical "domain of interest" in the $(x-t)$ plane is shown in Fig. 5 for $V = 0$ and $H = 0$. However, the same diagram would hold even if V and H were non-zero. Learn what is meant by "Dirichlet" boundary condition, and "Neumann" boundary condition. These are well known jargons.

Problems on pp.953-954: simple classification exercises.

- 3(a).
- 3(d).
- 3f
- 3g

I would like to supplement the book here by addressing the question of *uniquenss* of solutions. In English, the question is: how do I know there is only one solution to the problem as posed?

We shall deal with the simplest parabolic equation:

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

The domain of interest is $0 \geq x \geq \ell$ and $t \geq 0$ (into the future). The initial condition and boundary conditions for $u(x,t)$ are:

$$u(x,0) = I(x), \tag{2}$$

$$u(0,t) = a(t), \quad u(\ell,t) = b(t), \tag{3}$$

where $I(x)$, $a(t)$ and $b(t)$ are user-specified functions.

Suppose you and your roommate each found an exact solution to this problem, denoted as $u_1(x,t)$ and $u_2(x,t)$, respectively. Are they the same thing? Let us formally define $\phi(x,t)$ by:

$$\phi \equiv u_1 - u_2. \tag{4}$$

It is easy to be convinced that the PDE governing ϕ is:

$$\frac{\partial \phi}{\partial t} = \sigma \frac{\partial^2 \phi}{\partial x^2} \tag{5}$$

and the initial and boundary conditions are:

$$\phi(x,0) = 0, \tag{6}$$

$$\phi(0,t) = 0, \quad \phi(\ell,t) = 0. \tag{7}$$

The question of uniqueness of the original question boils down to: can we prove that ϕ is zero?

To proceed on the proof, we first suppose that ϕ is not identically zero (and later show that this supposition is untenable). If $\phi \neq 0$, we can multiply ϕ onto both sides of (5). Messing around a bit (watch for the details in the lecture), we can rewrite the result as follows:

$$\frac{1}{\sigma}\frac{\partial \phi^2/2}{\partial t} = \frac{\partial}{\partial x}\left(\phi\frac{\partial \phi}{\partial x}\right) - \left(\frac{\partial \phi}{\partial x}\right)^2. \tag{8}$$

Integrating the whole PDE, term by term, over $0 \geq x \geq \ell$, we obtain:

$$\frac{1}{\sigma}\frac{dE}{dt} = \left[\phi\frac{\partial \phi}{\partial x}\right]_{x=0}^{x=\ell} - \int_0^\ell \left(\frac{\partial \phi}{\partial x}\right)^2 dx \tag{9}$$

where

$$E(t) \equiv \int_0^\ell \frac{\phi^2(x,t)}{2} dx \geq 0. \tag{10}$$

The first term on the right hand side of (8) vanishes because ϕ is zero at the end points, while the second term is always non-positive. Thus, (8) can be rewritten as:

$$\frac{dE}{dt} = -\sigma \int_0^\ell \left(\frac{\partial \phi}{\partial x}\right)^2 dx. \tag{11}$$

Since $E(0) = 0$ and $E(t)$ is non-negative by definition, the only way (11) can be satisfied (when $\sigma > 0$) is $\phi = 0$ in the domain of interest.

In other words, we have just proved uniqueness for our simple parabolic problem when $\sigma > 0$. In real life problems, σ is always positive. That is why Greenberg set $\sigma = \alpha^2$ so that the positivity of σ is guaranteed.

What happens if you want to solve the parabolic problem for $0 \geq x \geq \ell$ and $t \leq 0$ (find out what happens in the past)? It is easy to show that this mathematical problem is considered ill-posed—-watch for the discussion in the lecture.

- §18.3, separation of variables. §18.3.1-§18.3.2 uses Fourier Series—because the PDE involved has constant coefficients, and the domain of interest permits separation of variables (what happens if ℓ is a function of t?).

Read example 5 on page 968-971 to see what happens for the case of a disk (instead of a rod). Fourier Series is no longer the best way to go! Bessel function of the first and second kind ($J_0(\kappa r)$ and $Y_0(\kappa r)$) are now the eigenfuntions, and κ is the eigenvalue. No "analytical" answers—as was the case when sine and cosines were the eigenfunctions—are available for the list of eigenvalues. Numerical tables for the Bessel (and other) functions are available in technical libraries, and many modern software packages have subroutines for them. Take a look at Fig. 8. How do you compute the "coefficients" of a Bessel function expansion of some (simple) $f(x)$? For Fourier Series, usually one can manage to do the needed integrations by using integration by parts and other tricks. For the general case, one needs computers and a good library of subroutines! This the main reason why most homeworks are limited to Fourier Series.

Problems on pp.974-980:

- #**9**(a,b,c,d). This is a straghtforward exercise. You need to make an intelligent decision on what you should choose as your fundamental period for your Fourier Series, and whether a Cosine or a Sine series would be best. After this, it is just a matter of computing the Fourier Coefficients to get the answer for (a). To get the rest of the answers, you need to make some judgment on how many terms you need to be concerned about. Remember, put the numbers in at the very, very last step.

- §18.4. Fourier and Laplace Transforms. It provides you with two examples: one you should use Fourier Transform (for x), the other you should use Laplace Transform (for t). For the problem assigned below, you are told to use Fourier Transform.

  Problem on p.989:

  - #**8**(b). Figure out your answer before peeking at the back of the book. The answer is obvious!
  - #**8**(c). This is a "show that" question, with the answer to be shown given.
  - #**10** Greenberg suggests that you use Fourier Transform. You may want to try Laplace Transform to realize that Greenberg

gave good advice! Just find the Fourier Transform of $u(x,t)$, *i.e.* $\hat{\omega}, t$, and don't worry about going through the chore of doing the inverse Fourier Transform.

- §18.5. Scan this section briefly. I will present the same material in an alternative manner in class.

- Read Chapter 17 Review at the end of the chapter, again.