

# MAE 306 Notes #5a

## Engineering Mathematics II

S. H. Lam

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### 1 Reading and Homework Assignments

Greenberg's book do not cover first order PDEs. This week, I will present materials on this topic, supplemented by notes given below. Problems are assigned at the end of the notes.

The problems are due on Tuesday, March 9nd, 1999, 5PM. Please submit your homework to the MAE 306 homework **IN** tray outside D-302, E.Q.

### 2 Supplementary Notes on First Order PDEs

The main reference for the notes that follows is: R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol.2, Chapter II; Interscience Publishers, John Wiley and Sons. 1961.

What happens when we have a first order PDE for a single dependent variable  $u$  and two independent variables  $x$  and  $y$ ?

#### 2.1 The Quasi-Linear Case

The most general quasi-linear first order PDE problem is:

$$A(x, y; u)p + B(x, y; u)q = R(x, y; u) \quad (1)$$

where

$$p \equiv \frac{\partial u}{\partial x} \quad (2a)$$

$$q \equiv \frac{\partial u}{\partial y}, \tag{2b}$$

and  $A$ ,  $B$  and  $R$  are given function of their arguments.

We use our pencil and draw a smooth line in the  $x, y$  plane. Let  $\sigma$  denoted a “coordinate” along the penciled line we drew (*e.g.*  $d\sigma^2 = dx^2 + dy^2$ ; but any other choice can be used). Now Consider two adjacent points  $d\sigma$  apart on this line—with the understanding that these two points are infinitesimally close. Let us denote  $x_2 - x_1$  by  $dx$ , and  $y_2 - y_1$  by  $dy$ . Since  $u(x, y)$ , we have by chain rule (read the following equation from right to left):

$$p \frac{dx}{d\sigma} + q \frac{dy}{d\sigma} = \frac{du}{d\sigma}. \tag{3}$$

Since both (1) and (3) must hold simultaneously, we obtain, by comparing the two equations:

$$\frac{dx}{d\sigma} = A(x, y; u), \tag{4a}$$

$$\frac{dy}{d\sigma} = B(x, y; u), \tag{4b}$$

$$\frac{du}{d\sigma} = R(x, y; u). \tag{4c}$$

Voila! You can poke the theory with a fork—it is done! We have reduced a PDE into a system of three ODEs. With a computer, this system of ODEs is readily solved (when the needed initial conditions are provided)!

So this is the situation. Given (1) and an *initial condition* line in the  $x, y$  plane on which the initial value of  $u$  is given. From each point on this initial condition line, we can use this system of three ODEs to compute for  $x(\sigma)$ ,  $y(\sigma)$  and  $u(\sigma)$ , the first two yields parametrically a “trajectory” in the  $x, y$  plane, and the value of  $u$  on this trajectory is parametrically known. The trajectory is called a *characteristics*. So each point on the initial condition line (which may be a finite line segment), a characteristic can be constructed on which the solution  $u$  is known.

If  $A$  and  $B$  do not depend on  $u$  (while  $R$  may still depend on  $u$ ), the characteristics can be computed and drawn without knowing the initial condition line or the values of  $u$  specified on it. Most importantly, at each point in space where  $A$  and  $B$  are not simultaneously zero, there is only one characteristic. Points at which  $A = B = 0$  are called singular points (You met them in MAE 305).

After you can computed from every point of a given finite segment of an initial condition line the associated characteristic, then the solution at any point which can be reached by a characteristic can be found. What about points which is not reached by any of these characteristics? The solution at these points are undertermined.

What happens if  $u$  is dependent on three independent variables  $(x, y, z)$ ? The complications to the computations of  $u(\sigma)$  is trivial.

As it stands, it appears that nonlinear quasi-linear problems and linear problems are not that different. From the point of view of solution procedure to get  $u(\sigma)$  on a selected characteristic, this is true. But from the point of view of finding  $u(x, y)$ , the quasi-linear case is much more complicated! Most importantly, quasi-linear problems (where either  $A$  and/or  $B$  depends on  $u$ ) can generate multi-value solutions for  $u$ . In other words, it is possible for a point in the  $x, y$  plane to be reached by several characteristic coming from several points on your initial condition line. This is not possible for linear problems. The waves on a sloping beach is governed by a quasi-linear equation, and the nonlinearity is responsible for the kind of waves that surfers love.

## 2.2 The Inviscid Burger's Equation

When  $A = 1$ ,  $B = u$  and  $R = 0$ , the problem is called Berger's equation, a most famous equation which exhibits the essential fancy properties of quasi-linear systems (in contrast to strictly linear systems).

## 2.3 Exercises

Solve the following problems, and graph the region where solution is found.

1 Consider

$$A = 1, B = 1, R = 0. \tag{5}$$

The initial condition is:

$$u = \sin(x) \quad \text{on the initial condition line} \quad |x| \leq \pi. \tag{6}$$

2 Consider

$$A = x, B = y, R = 1. \tag{7}$$

The initial condition is (in polar coordinates):

$$u = \sin(\theta) \quad \text{on the initial condition line } r = 1. \quad (8)$$

What is going on at the origin?

**3** Consider

$$A = y, B = -x, R = 1. \quad (9)$$

The initial condition is on the line  $y = 0$ :

$$u = \sin(x) \quad \text{on the initial condition line } |x| \leq \pi. \quad (10)$$

Is there a problem here?

**4** Consider

$$A = 1, B = u, R = 0. \quad (11)$$

The initial condition is on the line  $y = 0$ :

$$u = \cos(x) \quad \text{on the initial condition line } |x| \leq \pi/2 \quad (12a)$$

$$= 0, \quad |x| > \pi/2. \quad (12b)$$

Sketch the solution for different times. When does the solution become multi-valued?

### 3 Fully Nonlinear First Order PDEs

Instead of (1), we now consider the general first order PDE:

$$F(p, q, u, x, y) = 0, \quad (13)$$

where  $F$  is a differentiable single value function of its arguments. To make sure that (13) is a PDE and not simply an algebraic equation, we assume

$$F_p^2 + F_q^2 \neq 0 \quad (14)$$

where the subscripts denote partial derivatives. The previous case is thus seen as a special case—when  $F$  depends on  $p$  and  $q$  linearly. Now, no holds are barred!

First, let us look at a point in the three dimensional space of  $(u, x, y)$ . At this point, (13) tells us that  $p$  and  $q$  are related. Let us use  $\lambda$  as a parameter to parametrize this relation:

$$p = p(\lambda) \tag{15a}$$

$$q = q(\lambda) \tag{15b}$$

Let us differentiate (13) with respect to  $\lambda$ , holding  $(u, x, y)$  fixed. We obtain:

$$F_p \frac{dp}{d\lambda} + F_q \frac{dq}{d\lambda} = 0 \tag{16}$$

Of course, the chain rule result, (3), still holds. We differentiate (3) with respect to  $\lambda$  (holding  $u, x, y$  fixed) to obtain:

$$\frac{dp}{d\lambda} \frac{dx}{d\sigma} + \frac{dq}{d\lambda} \frac{dy}{d\sigma} = 0. \tag{17}$$

Comparing the last two equations, we obtain:

$$\frac{dx}{d\sigma} = F_p, \tag{18a}$$

$$\frac{dy}{d\sigma} = F_q. \tag{18b}$$

Using these in (3), we have:

$$\frac{du}{d\sigma} = pF_p + qF_q. \tag{19}$$

We can readily “touch base” by considering the special quasi-linear case to see that all results are identical to what were obtained precisely.

But, for the fully nonlinear case, the above system of three equations is not enough! This is because when  $F$  depends on  $p$  and  $q$  nonlinearly,  $F_p$  and  $F_q$  will depend on  $p$  and  $q$ . And at this point we do not know how  $p$  and  $q$  varies as we advance in  $\sigma$  along a characteristic. In other words, we need to find  $dp/d\sigma$  and  $dq/d\sigma$ . By chain rule, we have:

$$\frac{dp}{d\sigma} = \frac{\partial p}{\partial x} \frac{dx}{d\sigma} + \frac{\partial p}{\partial y} \frac{dy}{d\sigma}, \tag{20a}$$

$$\frac{dq}{d\sigma} = \frac{\partial q}{\partial x} \frac{dx}{d\sigma} + \frac{\partial q}{\partial y} \frac{dy}{d\sigma} \tag{20b}$$

To deal with these equations, we take partial differentiation of  $F$  with respect to  $x$  and  $y$ . We have:

$$\frac{\partial F}{\partial x} = F_p \frac{\partial p}{\partial x} + F_q \frac{\partial q}{\partial x} + F_u p + F_x = 0, \quad (21a)$$

$$\frac{\partial F}{\partial y} = F_p \frac{\partial p}{\partial y} + F_q \frac{\partial q}{\partial y} + F_u q + F_y = 0, \quad (21b)$$

Using (18a) and (18b) to eliminate  $F_p$  and  $F_q$ , we obtain:

$$\frac{\partial p}{\partial x} \frac{dx}{d\sigma} + \frac{\partial q}{\partial x} \frac{dy}{d\sigma} + F_u p + F_x = 0, \quad (22a)$$

$$\frac{\partial p}{\partial y} \frac{dx}{d\sigma} + \frac{\partial q}{\partial y} \frac{dy}{d\sigma} + F_u q + F_y = 0. \quad (22b)$$

Now, by the definition of  $p$  and  $q$ , we have:

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \quad (23)$$

Using (23), we find the the first two terms in the middle of (21a) and (21b) are simply  $dp/d\sigma$  and  $dq/d\sigma$ , respectively. Hence, we have:

$$\frac{dp}{d\sigma} = -(F_u p + F_x), \quad (24a)$$

$$\frac{dq}{d\sigma} = -(F_u q + F_y). \quad (24b)$$

Equations (18a), (18b), (19), (24a) and (24b) are five ODEs for the five unknowns,  $x(\sigma)$ ,  $y(\sigma)$ ,  $u(\sigma)$ ,  $p(\sigma)$ , and  $q(\sigma)$  on any characteristic of interest. The initial condition is obtained from the initial condition line.

### 3.1 The Huygen's Equation

Consider the case

$$F = p^2 + q^2 - 1. \quad (25)$$

The system of five ODEs for this problem is:

$$\frac{dx}{d\sigma} = 2p, \quad (26a)$$

$$\frac{dy}{d\sigma} = 2q, \quad (26b)$$

$$\frac{du}{d\sigma} = 2, \quad (26c)$$

$$\frac{dp}{d\sigma} = 0, \quad (26d)$$

$$\frac{dq}{d\sigma} = 0. \quad (26e)$$

Consider now a problem with the initial line given on  $y = \sin(x)$  with  $u(x, \sin(x)) = 1$  for  $|x| \leq \pi$ .

Let  $\xi$  be a marker on the initial condition line, defined by  $\xi = x$ . We have, by chain rule, along the initial condition line:

$$\frac{du}{d\xi} = p \frac{dx}{d\xi} + q \frac{dy}{d\xi} = 0. \quad (27)$$

It equals to zero because on the initial condition line  $u$  is a constant. Now, on the initial condition line we have  $\xi = x$  and  $y = \sin(x)$ . We thus have:

$$p(\xi) + q(\xi) \cos(\xi) = 0 \quad (28)$$

on the initial condition line. Since  $F = 0$  with  $F$  given (25) must be honored everywhere, we have:

$$q^2(\xi) (\cos^2(\xi) + 1) - 1 = 0, \quad (29)$$

which yields:

$$q(\xi) = \pm \sqrt{\frac{1}{\cos^2(\xi) + 1}} \quad (30)$$

and, with the help of (28),

$$p(\xi) = \mp \cos(\xi) \sqrt{\frac{1}{\cos^2(\xi) + 1}}. \quad (31)$$

Equations (31) and (30) provide the initial conditions for the integration of (26a)-(26e) on the characteristic coming from the characteristic located at  $x = \xi$  on the initial condition line (It should be “obvious” that the characteristic is perpendicular to the initial condition line). Since  $p$  and  $q$  are constants and  $u$  is linear with respect to  $\sigma$ , life is very simple indeed—until “cusps” shows up. Watch for this in the lecture.

## 3.2 Exercise

- 5 Discuss what goes on at the “ends” of the finite initial condition line for the above example.

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The bottom line is: first order PDE is a piece of cake—until cusps and multi-valued solutions show up.