MAE 306 Notes #6a
Engineering Mathematics II

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1 Mid-Term this Thursday

There is a mid-term this thursday, March 11, 1999. Appendix D, if needed in the mid-term, will be provided to you.

2 Study Hints, 3/8/99.

Chapter 17, Fourier Series, Fourier Integral, and Fourier Transform. [And Notes 1a and 1b.]

- Here, the set of "basis functions" are sines and cosines. The fact that they satisfy (24a,b,c) on page 857 are information provided without comments. Then the big deal is: given any piece-wise continuous function in the domain $(-\ell \leq x \leq \ell)$ of interest, we can find its Fourier Series Representation.
- You can use the full set of sines and cosines, or you can just just the sines or just the cosines—by extending the domain of interest.
- You should know how to decompose any function into the sum of an even and an odd function.
- You should know what is Gibb's phenomenon.
- On Fourier Integral: the final formula is (2a) and (2b) on page 914-915. You don't need to remember the formula (*i.e.* the factor in front of the integral). I will give it to you if you should need it.

- On Fourier Transform. the final formula is (6a) and (6b) on page 921. You don't need to remember the formula (*i.e.* the factor in front of the integral). I will give it to you if you should need it.
- You must know what a Dirac Delta function is, and its Fourier Transform.
- You should know the following identity by heart (see Notes 2a):

$$\exp(i\xi) = \cos(\xi) + i\sin(\xi). \tag{1}$$

- Notes 1b tells you that the Fourier Series is just one example of a class of basis functions. The concept of inner product of two functions is introduced. The concept of a linear second order differential operator $\mathcal{L}$ is introduced, and the derivation of the *adjoint operator* $\mathcal{L}_*$ should be understood. When $\mathcal{L} = \mathcal{L}_*$, the operator is said to be adjoint. The eigen stuffs associated with the (self-adjoint) operator $\mathcal{L}$ are introduced. What did we learn from all this stuff? We find out that the set of eigen functions can be ordered by the magnitude of their eigenvalues which has a minimum and has no maximum (its maximum is infinity). We find out that these eigenfuctions are orthogonal to each other. We find that if more terms are included in a representation, the "better" is the representation in the mean square error sense—provided we do not skip over any eigenfunctions. So, if the second order linear differential operator of interest is self-adjoint but does not have constant coefficients, the recommended basis function would be the eigenfunctions of the operator in question, and not the Fourier Series. We also learn how to numerically find the eigenfunctions in ascending order of its eigenvalue.

Chapter 18, Diffusion Equation :

- First, we did the classifications. You should understand how $B^2 - AC$ becomes the formula to reckon with in classification of second order quasi-linear PDEs. Of course, you are expected to know what is a quasi-linear differential equation (ODE or PDE): the highest order derivative occurs linearly.
- I would like you to be able to reproduce the uniqueness proof in Notes 3a.

- On Green's function: I want you to understand what a Green's function is and what it does for us. Note that G satisfies (16) on page 4 of Notes 3b. Do you see why the sign of the time derivative term is changed? Note that we require $G = 0$ on the boundaries. Do you know why? The bottom line is (21a,b,c,d) on page 5 of Notes 3b; it gives you explicit formula for the effects of initial condition, source term and boundary conditions.
- I told you in class that the "fundamental Green's function," G_∞, can readily be found by Fourier Transform. I may very well ask you to do this for mid-term (you are supposed to know what the Fourier Transform of a Dirac Delta function is).
- In all of our class work, we had assume the values of the unknown u at the boundaries are known. Suppose we have $u(x = 0; t) = 0$ and $\partial u/\partial x(x = \ell, t) = 0$ as our boundary conditions. Can you handle such boundary conditions?
- Just be aware of the method of similar solutions. If the problem posed has no characteristic time or length, then the method of similar solution is a cheap way to go.

Notes 4a, Numerical Methods :

- The spatial domain is discretized into nodes. The size of the nodes (the distance between nodes) is decided by your need for resolution of your computed solutions. If you are interested in $0 \le x \le 1$, and you want a data point every 0.01 apart, you need 100 nodes. If you are a cheapie and allow only three nodes, you will get a lousy solution.
- The new revelation is: a parabolic PDE, after discretization, is just an algebraic matrix equation! If forward difference is used, the matrix involved is the simplest kind—it is diagonal! If Crank-Nicholson is used, we have a tridiagonal matrix.
- The big deal is that for the explicit scheme, the dimensionless parameter r must be less than or equal to 0.5, limiting the size of time step you may use. The Crank-Nicholson scheme removes this stability restriction of the size of the time step. But you must verify by computation that reducing your last used time step did

not change your solution significantly. Otherwise, your time step is too big.

- Once you learn how to invert a "sparse" matrix (remember, NO respectible person would ever admit that he called the inverse subroutine for a big, sparse matrix), complications that would normally ruin analytical methods are easily handled by numerical methods.

Notes 5a, First Order PDEs :

- Must understand the concept of characteristics. All first order PDEs (for ONE dependent variable) are system of ODEs in disquise. Solutions are allowed to be "discontinuous" across characteristics—by a bit of intellectual cheating.
- The strictly linear case: its a piece of cake. The big deal is: where in the x, y plane does your solution apply?
- The quasi-linear case: its strightforward to get the solution. The big deal is: is the solution multi-valued? Of course, it continues to be important to know where in the x, y plane does your solution apply (for your initial condition).
- Fully nonlinear first order PDEs. Slightly more complicated—you now need to compute p and q along your characteristic. You must be aware of the possibility of multi-value solutions.

2.1 Some useful methodology for first order PDEs

Consider

$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3. \tag{2}$$

The initial condition line is

$$u = 1.2\sin(y), \qquad x = 0; \quad |y| \leq 1. \tag{3}$$

We use ξ as a marker on the initial line, defined by $\xi = y$.

This is a linear first order PDE. Integrating the three ODEs and honoring the initial conditions on the initial condition line, we obtain:

$$x = \sigma, \tag{4}$$
$$y = \xi \exp(\sigma), \tag{5}$$
$$u = 1.2 \sin \xi + 3\sigma. \tag{6}$$

Here, σ is the coordinate along the characteristic issuing from the initial condition line.

The following is the usual advice on what to do to find $u(x, y)$:

- Eliminate ξ between (4) and (5) to obtain σ as a function of x and y.
- Eliminate σ between (4) and (5) to obtain ξ as a function of x and y. For fixed ξ, this gives you the equation of the characteristic.
- Plug them into (6)

For the problem at hand, we have:

$$\sigma = x, \tag{7}$$
$$\xi = y \exp(-x). \tag{8}$$

Hence, the solution for $u(x, y)$ is:

$$u(x, y) = 1.2 \sin(y \exp(-x)) + 3x. \tag{9}$$

Equation (9) as it stands is correct and is extremely dangerous! There is no warning that it is valid only in the region where the characteristics issuing from the initial condition line can reach. For example, (9) would not complain if it is used to find $u(-4, 8)$, but the answer obtained is TOTALLY meaningless.

Please check the course web page wednesday night to see if I have updated this. I am posting it now to allow some of you to take a look before tuesday's class. And I will print and distribute it in tuesday's class. But I may want to make some minor changes later.

See you in tuesday's class.