MAE 306 Notes #7a
Engineering Mathematics II

S. H. Lam

March 25, 1999

1 Reading and Homework Assignments

Study Chapter 19, Wave Equations, pp. 1017-1055 of Michael Greenberg's *Advanced Engineering Mathematics*, and do the problem assigned below.

The problems are due on Tuesday, March 30th, 1999, 5PM. Please submit your homework to the MAE 306 homework **IN** tray outside D-302, E.Q.

2 Separation of Variables

Greenberg's treatment of wave equations consists of separation of variables, the D'Alembert general solution for one dimensional unsteady waves, and an optional treatment by Fourier Transform (Laplace Transform was declared "messy" by Greenberg), again in one spatial dimension (plus time).

The methodology of separation of variables works for the wave equation, proddingly and mechanically, in essentially the same way. If you have a rectangular, circular, spherical (or any one of the 16 "separable coordinate systems for the Laplacian) physical domain of interest, you can use the method of separation of variables. This method will methodically get you the solution in terms of a bunch of infinite series. The only "new twist" is that the $T(t)$ ODE is now second order, and now you have TWO integration constants. The initial condition now requires that u and $\partial u/\partial t$ be specified on the initial condition line (or surface).

2.1 Problems

5b,d , page 1031. This is totally straightforward. Use equation (5.1), ignore question 5a.

1d , page 1042. Again, totally straightforward.

2a , page 1053. Just to make sure you know the Chain Rule.

7a,b , page 1053. Do not peek at the answer at the back of the book before you are convinced that you have gotten the correct answer for 7a.

3 Supplementary Notes on Wave Equations

But, wave equations have fascinating properties which is not easily discernable from separation of variables. I hope you enjoy the supplementary materials given below.

We write down the generic second order linear wave equation as follows:

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \tag{1}$$

where a has the dimension of velocity and is the "wave speed" while the Laplacian in Cartesian coordinates is:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2}$$

When we are dealing with a one-dimensional unsteady wave problem, $u(x,t)$, we have a two independent variables problem. We can go look for characteristics, and we will find two real characteristics (lines): $x - at = R$ and $x + at = L$ where R and L are constants on the respective characteristics. When we deal with more than one-dimensional $u(x,y,t)$ or $u(x,y,z,t)$, we need to think about characteristic *surfaces*. Pragmatically, the method of characteristics (lines) is a useful method in one-dimensional waves. No one has developed a method of characteristics for characteristic surfaces—because it is expected to be messy and non-competitive with other available methods. Nevertheless, the concept of characteristics remains valid in multi-dimensional problems.

3.1 Initial and Boundary Conditions

What kind of domain of interest in (x, y, z, t) space are we normally interested? We know t is a distinct independent variable—it is different: $+t$ points to the future, while $-t$ points to the past. If something happens at t_*, its consequence should exist only for $t \geq t_*$. The self-evident requirement is called *causality* requirement. It must be kept in mind, because the equation itself says if you replace t by $-t$, the new equation looks the same as before. Thus the equation can take any solution you found, reverse its time, and tell you that the new result is also a solution to the PDE. While this claim is true, the new solution does not satisfy the casuality requirement and must therefore be rejected (if you care about your solutions making physical sense).

On the "boundary" of your physical (x, y, x) region, you need "boundary conditions": u or the "normal component" of ∇u (or a linear combination of these two) should be specified. At $t = 0$, both u and $\partial u/\partial t$ are required.

What happens when your domain of interest in $(x, y, z; t)$ space is a smooth domain with no sharp distinction between the initial condition portion and the boundary portion (think of a round bottom tub in $(x, y; t)$ space, with the bottom of the tub located a $t = 0$). What do we do with boundary and initial conditions on the round part of the tub?

Here is the general answer: when characteristics exists, the number of conditions needed at the surface of the domain of interest (in x, y, z, t space) equals the number of chacteristics entering the domain (with the time arrow pointing toward the future).

What happens if all the characteristics are leaving the domain of interest at the surface? No conditon is then allowed at that surface.

4 Response to a Delta Function Pulse

Consider the following problem:

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0 \tag{3}$$

where a is a constant with dimension of velocity and $\nabla^2 u$ is the Laplacian of u.

4.1 The One-Dimensional Case

The PDE is:

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{4}$$

Consider the problem of $u(x,t)$ with the following initial condition:

$$u(x,0) = \delta(x), \quad \frac{du}{dt}(x,0) = 0, \tag{5}$$

where $\delta(x)$ is our old friend, the Dirac Delta function. The D'Alembert general solution is:

$$u(x,t) = \frac{1}{2}\left(\delta(x-at) + \delta(x+at)\right). \tag{6}$$

You can check that it is indeed a solution (plugging it into the original PDE, pretending that the Dirac Delta function is differentiable, and checking the initial conditions).

So, a delta function at the origin is divided into two delta functions, one runs to the left, and one runs to the right, with velocity a. After the delta function passes, all is quiet.

4.2 The Two-Dimensional Axisymmetric Case

The PDE is (in polar coordinates):

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right). \tag{7}$$

The initial condition is: u is a Dirac Delta function at the physical origin at $t = 0$, and du/dt is initially zero. The solution is:

$$u(r,t) = \frac{1}{2\pi\sqrt{a^2t^2 - r^2}}, \quad r^2 \leq a^2t^2, \quad t \geq 0, \tag{8}$$

and is zero elsewhere. So, a delta function at the origin causes a cylindrical wave to propagate away from the origin, with velocity a. After the "singularity" passes, there is a "tail" to the disturbance.

4.3 The Three-Dimensional Axisymmetric Case

The PDE is (in spherical coordinates):

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right). \tag{9}$$

The initial condition is: u is a Dirac Delta function at the physical origin at $t = 0$, and du/dt is initially zero. The solution is:

$$u(r,t) = \frac{\delta(r - at)}{4\pi r}. \tag{10}$$

So, a delta function at the origin causes a spherical wave to propagate away from the origin, with velocity a. After the "singularity" passes, all is quiet again. We should be grateful that we live in a three-dimensional world! If we live in a two dimensional world, it would be difficult to talk to each other, and music would be difficult to make!

See Notes 7b for additional supplementary materials.