MAE 306 Notes #7b
Engineering Mathematics II

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1 More Supplementary Materials

On wave equation, you are required to be fully competent to do separation of variables. For linear problems, the standard trick is to get rid of non-zero boundary conditions by hook or by crook, letting the "initial condition" and the source term in the original PDE to take on all the burdens. Then solve the problem with homogeneous boundary conditions using separation of variables, mindful of the fact that the spatial solutions are eigenfunctions of a Sturm-Liouville problem (provided you checked) so that they are orthogonal basis functions.

The details of the supplementary materials below are supplementary; the concepts must be understood.

2 The Concept of Characteristics

When we started the course, we did a classification study of quasi-linear second order PDEs, and came up with the $B^2 - AC$ stuff. Lets revisit the ideas there again, for the one-dimensional unsteady wave equation:

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

We introduce p and q by:

$$p = \frac{\partial u}{\partial t}, \tag{2}$$

$$q = \frac{\partial u}{\partial x}. \tag{3}$$

Consider now a smooth "initial condition" line, drawn arbitrary by you, on the x, t plane. On this line, the values of u, p and q are specified, and they must satisfy the chain rule:

$$du = pdt + qdx \tag{4}$$

where du, dx and dt are changes of u, x, t between two infinitesimally-close points on the initial line. If we use ξ as a "marker" on this initial condition line, then $u(\xi)$, $p(\xi)$ and $q(\xi)$ are known. The question is: can we compute the value of u, p and q on an adjacent line (infinitesimally close) to this initial condition line. Now, it is totally straight forward to get u on this (infinitesimally-close) adjacent line—since u, p and q are known. The critical issue is how do we get the new p and q on this (infinitesimally-close) adjacent line. If we can, then we can, in principle, "march" away from the initial condition line to cover more real estate on the x, t plane.

We have four differential equations:

$$\frac{1}{a^2}\frac{\partial p}{\partial t} + 0 \quad + 0 \quad - \frac{\partial q}{\partial x} = 0, \tag{5}$$

$$0 \quad + \frac{\partial p}{\partial x} - \frac{\partial q}{\partial t} + 0 = 0, \tag{6}$$

$$\frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}dx + \quad 0 \quad + 0 = dp, \tag{7}$$

$$0 + \quad 0 + \frac{\partial q}{\partial t}dt + \frac{\partial q}{\partial x}dx = dq. \tag{8}$$

The first equation is the original wave equation. The second equation is an identity—assuming that u is twice differentiable. The third and fourth equations relate the changes of p and q between two points on the initial value line infinitesimal distance apart—by chain rule. We now have four equations for the four "unknowns," the partial derivatives of p and q with respect to t and x. These equations can be rewritten as:

$$\mathbf{M} \cdot \mathbf{X} = \mathbf{b} \tag{9}$$

where

$$\mathbf{M} = \begin{vmatrix} \frac{1}{a^2} & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{vmatrix}, \quad \mathbf{X} = \begin{vmatrix} \frac{\partial p}{\partial t} \\ \frac{\partial p}{\partial x} \\ \frac{\partial q}{\partial t} \\ \frac{\partial q}{\partial x} \end{vmatrix}, \quad \mathbf{b} = \begin{vmatrix} 0 \\ 0 \\ dp \\ dq \end{vmatrix}. \tag{10}$$

If $\mathbf{M}$ is non-singular, we can simply solve this equation for $\mathbf{X}$, allowing us to "march" p and q from the initial line to the infinitesimally-close) adjacent line. In principle, then, we can loop and march on and on.

But what happens if $\mathbf{M}$ is singular?

Since $\mathbf{M}$ depends on the "geometry" of the initial line, the question then reduces to: what is the relation between dx and dt that can make $\mathbf{M}$ singular? This was the logic that was used previously to deduce the $B^2 - AC$ criteria.

For the PDE at hand, the $B^2 - AC$ criteria says it is hyperbolic. Setting the determinant of $\mathbf{M}$ to zero, we find there are TWO characteristics, defined by:

$$\frac{d_{\pm}x}{dt_{\pm}} = \pm a \tag{11}$$

The notation should be obvious: $d_+/dt_+ = a$ is a right-running characteristic, and $d_-/dt_- = -a$ is a left-running characteristic. If a line drawn (by you) on the x, t plane is a characteristic, it is NOT eligible to be an initial condition line! Why? Because, by Kramer's rule, the values of dp and dq along this line (between two infinitesimally-close points) must satisfy a "compatibility condition." Setting the numerirator of the Kramer's rule formula for $\mathbf{X}$ to zero, we can obtain the so-called "characteristic relations." In our case, these relations are $d_{\pm}p \pm a d_{\pm}q = 0$.

If you review the derivations, you will find the same characteristics if a is not a constant, but depends on u, x, t, p and q, and if (1) contain any additional terms which do not involve the highest derivatives (quasi-linear).[1] For our present exposition, we shall assume a to be a constant. We can then introduce the new independent variables:

$$\xi = x - at, \tag{12}$$
$$\eta = x + at. \tag{13}$$

The right-running characteristic has ξ =constant, with η serving as a marker. The left-running characteristic as η =constant, with ξ serving as a marker. If you work out the details (taking full advantage of a=constant), you will find $d_{\pm}u = 0$ (no source term). In principle, then, the solution of our original (1) can be constructed, either analytically or numerically, point by point—once the initial line and the boundary lines are given and the appropriate initial and boundary conditions are given.

[1] The characteristic relations will be affected by the "source term."

Normally, the initial line for a x, t problem is specificed by $t = t_o$ =constant. The boundary lines are usually specified by $x = L$ and $x = R$ where L and R are constants. For such problems, we need both u and p on the initial line, and u or q (or a linear combination) on the boundary line. What happens if the initial and boundary lines are given by a continuous curve? What happens if the boundary line is moving—when L and R are functions of time?

The general rule is: the number of conditions needed on a line equals to the number of characteristics entering the region of interest in forward time. We will talk about them in class.

3 Numerical Methods

If you discretize as we did for the diffusion equation, you will derive an algebraic equation for your discretized unknown, $u(m, n)$ where m and n are your spatial and time "indices." The spatial grid points are Δx apart, and the time-wise spacing is Δt about. To march forward in time by Δt, you go from n to $n + 1$.

You may recall in the diffusion equation, the value of Δt for the explicit algorithm has an upper bound. The same thing is true for the wave equation. Let c denoted the maximum "speed" of the characteristics (for (1), $c = a$). For explicit algorithms, the following condition must be satisfied:

$$\frac{c\Delta t}{\Delta x} \leq 1. \tag{14}$$

The reason why should be intuitively obvious. If not, ask me in class.